GRAVITATION - RELATIVITY - GEOMETRY*

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Dr. Prahalad Chunnilal Vaidya was an eminent Physicist and Mathematician. He made significant contributions in the field of relativity and gravitation. Through his research work in the general theory of relativity, he gave the Vaidya Metric. Vaidya was also a follower of Gandhian philosophy and an Educationist.

The paper explores the universality of the laws of physics. The profound impact of Newton's law of universal gravitation and its foundational role in modern science and engineering is discussed. It examines the universality of the Newtons law highlighting its applicability beyond the Earth, similar to other natural laws such as Boyle's law. The paper introduced the principle of relativity, which is a criterion for distinguishing universal laws from local effects. Through the example of Einstein's thought experiments, the paper discusses the scenario of a freely falling lift to illustrate the relativity criterion and test the universality of Newton's law of gravitation. Through these discussions, the paper highlights the significance of Newtonian mechanics and the principle of relativity in our understanding of natural phenomena. The discussion transitions to the problem of geometry, emphasising that the description of gravitational phenomena is linked to the Geometry used.

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It is said that when Archimedes sat in his bathtub, modern science was born and when Newton sat in his orchard it came of age. Newton reorganised the then existing knowledge about motion of bodies into his three famous laws of motion and made the most capital discovery of the law of universal gravitation. His law of gravitation may be stated in the following form:

> "Every particle in the Universe attracts every other particle with a force which varies directly as the product of the masses and inversely as the square of the distance between them".

It is well known that with the law of gravitation and with his three laws of motion, Newton could explain diverse phenomena like the motion of a pendulum or of a spinning top, the motion of the planets, the phenomena of tides and so on. As a matter of fact the entire edifice of modern engineering stands essentially on the three laws of Newton. Let us examine his law of gravity in detail.

Universality of the Law

Newton's law is called the '*Law of Universal Gravitation*.' In the statement of the law which we have given above there is a mention about something that happens with any two particles in the Universe. One may as well wonder: What led Newton to describe something which should obtain any where in the Universe when his observations were limited to his orchard only? This is a question which can be asked about all laws of Physics. Boyle performed a set of experiments in

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his kitchen and deduced what is known as Boyle's law that at constant temperature, the pressure of a gas varies inversely as its volume. Nobody has ever suggested that this law will be true only if the experiments were performed in Boyle's kitchen. This points to an essential difference between local or municipal laws and the laws of nature. Laws of nature, as the name suggests, are not laws of a particular place or time but must hold good everywhere in nature.

How does a scientist deduce a law of nature holding good anywhere in the Universe from a set of observations limited to his laboratory? Scientists must have before them some criteria by which they must be able to weed out local effects in their observations, single out universal effects and weld these universal effects into a law of nature. Such a criterion must have existed in some form or another since the days of Archimedes when modern science was born.

Such a criterion is known by the name of the *Principle of Relativity*. A principle of relativity has been known since the days of Archimedes, but the first conscious use of this principle for testing laws of nature was made explicitly by Einstein only in 1905, and it is no wonder that after that date there has been a phenomenal rise in the growth of our knowledge about nature.

Relativity Criterion

We shall now try to understand this criterion of relativity and then see how Einstein used it to test the universality of Newton's law of gravitation.

The physics that is studied in India is the same as the physics that is studied in the USA or USSR. The experiments that are described in a textbook of physics could be performed anywhere on the Earth and if the experiments are performed under prescribed conditions, the same results will follow everywhere. But what about the Moon? If the Boyle's law apparatus is transferred to the Moon or to the planet Mars and a constant temperature for the gas is maintained, do we expect the pressure and the volume to vary along the familiar hyperbola? We should really expect the same behavior but we are not that sure for the Moon or Mars as we were for any laboratory on Earth. Let us be more explicit. We have here two comparisons: (i) comparison between results of identical experiments performed at a laboratory, say 'I', in India and a laboratory, say 'U' in USA. (ii) a comparison between experiments performed in a laboratory 'I' on the Earth and a laboratory 'M' on the Moon. One feels quite sure about obtaining identical results from identical experiments performed in 'I' and 'U', but one is not that sure about the same experiments when performed in 'I' and 'M'. What is the difference between the pair I and U and the pair 'I' and 'M'? Of course 'I' and 'U' are both fixed on the Earth while I is on the earth and 'M' is on the moon and the moon moves round the Earth. One essential difference which leads to the doubt expressed above is the difference of relative motion. 'I' and 'U', have no relative motion but 'I' and 'M' have relative motion. If we can ensure that relative motion between observers will not affect their observations, we can be sure of the identity of results at 'I' and 'M'. The essential feature of the criterion of relativity is to ensure that the relative motion between two observers should not affect their observations of nature. We can state this criterion of relativity in a more refined way as follows:

If two observers having motion relative to each other describe a certain phenomenon in identical terms then that phenomenon is a phenomenon of nature and the identical description is a Law of Nature.

Einstein's Thought Experiments

We shall now see how the above criterion of relativity was used by Einstein to test whether Newton's law of gravitation was really a law of nature or not.

Many must have observed falling apples before the days of Newton. But when Newton observed the falling apple in his orchard, he was conscious that he was observing a phenomenon of nature. Yes, a phenomenon of nature in the sense of the definition given here in the earlier section. This was the reason why from this simple observation, he was led to a law of nature. Einstein thought out an experiment to test whether this phenomenon of falling apple satisfied the criterion of relativity in order to be classified as a phenomenon of nature.

Let us think of a lift descending from the top floor of a high skyscraper like the Empire State Building and consider the unfortunate event in which the cable supporting the lift snaps. The lift will then start falling freely under gravity like the celebrated apple of Newton's orchard. Let us further imagine a mathematician 'A' in this freely falling lift. He can, of course calculate the time 't' (which will be of the order of 2 or 3 seconds for most skyscrapers) which the lift will take to reach the ground. While we, on the ground are worried about what would happen after that time 't', the Mathematician 'A', being a wise man, would like to make the best use of these 2 or 3 seconds available to him and so decide to test Newton's Law of Gravitation. In order to make a comparison we must have another observer 'B' on the ground. Since ordinary persons on the ground would at that moment be thinking about police, ambulance, hospital and such other things, we must imagine 'B' to be a mathematician on the ground who is not normally worried about such mundane things. (It is no wonder that the experiment is a thought experiment i.e., an experiment which is imagined but not actually performed. Who would like to take the place of the wise man 'A'?)

So 'A' decides to test gravity. He takes out a chalk piece from his pocket, raises it to the level of his eyes and then lets it go. What will he observe? Will he see the piece of chalk moving down towards the floor of the lift? How could that happen? He himself is falling freely i.e., his eyes have the same speed and acceleration as this piece of chalk. As the piece of chalk falls down, he with his eyes also falls down the same distance. Thus the vertical distance between his eyes and the chalk piece does not change. He will find the piece of chalk poised in the air before his eyes. For him, the piece of chalk is not falling downwards but remains at rest at the level of his eyes. He may as well be wondering: "Has the earth gone on vacation? Why is it not attracting this piece of chalk?" Earth is a body in the universe. The piece of chalk is also a body in the universe. For the observer 'A' these two bodies in the universe do not attract each other.

But the other mathematician on the ground maintains that they do attract each other. For B, the piece of chalk is falling downwards towards the earth. For him the lift and therefore the observer 'A' is also falling freely downwards towards the Earth. This means that the two observers 'A' and 'B' have relative motion. We can therefore imagine two observers in relative motion such that the two describe the phenomena of falling objects in different terms. 'A' says objects are not falling, 'B' says they are falling. Thus the phenomenon of falling apples is not a natural phenomenon in terms of our definition. The statement of the law of gravitation as given by Newton does not pass the test of relativity criterion as illustrated by this thought experiment of Einstein.

So what! Does this mean that gravitation does not exist? Well, one has just to throw a stone vertically upwards and not move away from the place of projection to receive a direct proof on the head that gravitation does exist! (This should again be taken as a thought proof—proof to be thought of but not to be demonstrated!)

This only means that the law of gravitation has to be reformulated in such a way that both 'A' and 'B' describe gravitational phenomena in identical terms.

Problem of Geometry

Let us turn again to Einstein's thought experiment of the falling lift, in order to see that the problem of reformulation of the law of gravitation as referred to at the end of the last section is essentially a problem of geometry.

We fix our attention on the observer 'A'. He observes the piece of chalk poised in the air in front of his eyes. Let us imagine that 'A' now gives a tick to the chalk in the horizontal direction away from his eyes. He will then see that the chalk piece sails uniformly in a straight line away from him and strikes the opposite wall of the lift at a point on the same level as his eyes.

Let us next go to observer 'B' and find out what he has recorded. 'B' has found that the chalk piece has moved onwards as well as downwards. He has thus observed the path to be a curved line, in fact, a parabola.

It is now our turn to wonder. Is the chalk really moving in a straight line or in a parabola?

The observer 'B' who experiences gravitation in the Newtonian way finds paths of certain objects to be parabolas. The observer 'A' who does not experience gravitation in the Newtonion way finds the paths of the same objects to be straight lines. The difference in the observations of gravitational phenomena, by the two observers is also reflected in a difference in the geometrical interpretation of curves by the two observers. It appears therefore, that the observation of a gravitational situation is, in some intricate way, related to the geometry used to describe the situation. We are searching for a description of the law of gravitation which would be identical for both observers 'A' and 'B'. Can we not do it by trying to fix up the identity of the geometry used by both 'A 'and 'B' for describing gravitational phenomena? We can; and that is what Einstein has actually done.

In what follows we shall therefore go into the details of development of different geometries to see how such a programme of formulation of the relativistically invariant gravitational theory was successfully was completed by Einstein.

Euclidean Geometry

The geometry that is studied in schools is the Euclidean Geometry. Some two thousand years back the Greek geometer Euclid collected all the knowledge of geometry existing at his time, organised it and wrote down the oldest known scientific textbooks Euclid's Elements. These books have continued to be in use in more or less unchanged form all these years. But the geometry described in these Elements would not be called Euclidean geometry simply because Euclid collected the existing knowledge and presented it in an organised form. Euclid did something more than that. He looked into the large mass of theorems of geometry and found that all these theorems could be derived from a set of axioms. These axioms, in their turn, could not be 'derived' or 'proved' but had to be assumed at the start. Euclid enumerated five unproved axioms at the beginning of his Elements and then gave logical deductions of various theorems from these axioms. He very well knew that the entire edifice of geometry that he presented in his Elements ultimately depended on the choice of the five axioms. This is the reason why the logical system of geometry based on the axioms chosen by Euclid is given the name of Euclidean Geometry.

The geometry that we study in schools is Euclidean geometry. A question naturally arises. Are there geometries which are not Euclidean? What would happen if we discard the five postulates of Euclid and choose some other axioms in their place? Well, one thing is certain—choosing another set of axioms will not be enough. One must be able to use these axioms consistently and deduce from them theorems which are not mutually contradictory. We shall now try to understand two such successful attempts at discovering non-Euclidean geometries.

The Parallel Postulate

One of the five axioms of Newton is the following one which has come to be known as the parallel postulate:

> "Given a straight line and a point outside it one and only one straight line can be drawn parallel to the given line and passing through the given point."

Euclid himself was not happy with his postulate. He thought that with the help of the other four axioms and with the help of theorems derived from these four axioms it would be possible to prove this parallel postulate as a theorem. He made many attempts in this direction but failed. And then only, almost reluctantly, he admitted this postulate in the list of his unproved axioms. For almost two thousand years after Euclid this particular postulate has been the subject of study by many geometers. Many tried to replace this postulate by another one and develop a self-consistent geometry. The first such successful attempt was that of the mathematician Lobatchewsky in 1820's. He kept the first four axioms of Euclid unchanged but replaced the fifth one by the following;

> "Given a straight line and a point outside it at least two straight lines can be drawn parallel to the given line and passing through the given point."

Lobatchewsky was able to develop a consistent geometry based on these postulates. One of the first propositions of his geometry was: "Given a straight line 'l' and a point 'O' outside it, if two parallel straight lines can be drawn through 'O' parallel to 'l', then three such lines can be drawn through 'O'.

If three parallel lines can be drawn through 'O' parallel to 'l', then four such lines can be drawn and so on. Thus from an outside point 'O' an infinite number of straight lines can be drawn parallel to a given line 'l'."

It is not very difficult to realise that Lobatchowskian geometry will be quite different from Euclidean geometry. As an illustration of this difference let us take the well known theorem about the sum of the three angles of a triangle. In Euclidean geometry this sum has two right angles. If we try to recapitulate the proof of this theorem of Euclidean geometry we shall see that the proof depends on producing BC to E and drawing CD parallel to AB. (see Fig. 1).

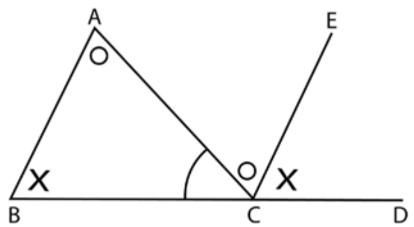

Fig. 1

Now AB is a straight line and C is an outside point. In Euclidean geometry from C only one straight line CD can be drawn parallel to AB and so the familiar proof of this theorem could be given. But what would happen to this proof if from C two straight lines could be drawn parallel to AB?

In Lobatchewskian geometry the corresponding proposition for the sum of the three angles of a triangle is: The sum of the three angles of a triangle is always less than two right angles.

At this stage one may venture a question. Which of the two geometries is the correct geometry? But, wait a minute. What does one mean by correct geometry? Let us remember that what we are discussing at present is mathematics and not physics. There are no laboratory tests for propositions of mathematics. Propositions of mathematics are self-consistent creations of human mind based on a set of assumed results, a set of accepted procedures and that queer abstraction of common sense known as logic. A triangle of geometry cannot be constructed in a workshop. It can only be constructed by geometrical (not physical) constructions. In that sense any self-consistent system of geometry is as correct as any other.

A more sensible query about these geometries would be the following: Which of the two geometries is more useful? The ultimate aim of this article is to answer this query. So let us wait till we reach the end of our discussions.

Riemannian Geometry

We turn again to the parallel postulate. 'l' is a given line and 'O' is a given point outside it. Euclid assumed that only one straight line can be drawn through 'O' parallel to 'l'. Lobatchewsky assumed that more than one straight line can be drawn through 'O' parallel to 'l'. One case was still left for consideration. As late as the middle of the nineteenth century, Riemann accepted the first four axioms of Euclid but replaced his fifth postulate by the following:

> "Given a straight line and a point outside it, no straight line can be drawn parallel to the given line through the given point".

He was able to construct a self consistent logical system of geometry based on these postulates. Thus was constructed the third geometrical system known as Riemannian geometry. In Riemannian geometry, there are not parallel straight lines. In other words, all straight lines must be intersecting straight lines.

To realise how completely different this geometry is from the familiar Euclidean geometry or the unfamiliar Lobatchewskian geometry, let us consider the triangle-theorem in this geometry. In Riemannian geometry the sum of the three angles of a triangle is always greater than two right angles.

And yet we who live on the surface of the Earth should not find anything new in Riemannian geometry because the geometry on the surface of a sphere cannot be Euclidean but must be Riemannian.

By definition, a straight line joining two points is the shortest distance between these two points. Now the shortest distance between two points on a sphere is along the arc of the great circle joining the two points. Hence a 'straight' line on a sphere is a great circle. Now, it is not possible to draw a great circle parallel to a given great circle. One can draw a parallel small circle but not a parallel great circle. Thus if one tries to draw a parallel 'line' one does not get a great circle i.e., one does not get a 'straight' line. Any two great circles on a sphere must intersect. There are no parallel great circles on a sphere and therefore there are no parallel straight lines in the geometry on a sphere. This geometry must therefore be Riemannian.

How is it then, that we have been freely using Euclidean geometry in surveying and other practical measurements of distances on the surface of the earth? The football field on the Earth is a plane and we draw straight lines as 'straight' and not curved like great circles. As a matter of fact, we use Euclidean geometry on the football field. But a football field is a very small—indeed insignificantly small—portion of the surface of the Earth. On the other hand when the path of a non-stop airplane flight between two distant points on the Earth is plotted, this path is invariably a great circle. The curvature of the sphere can be neglected when we consider only a small portion of the surface of the sphere and so this small portion could be regarded as flat and great circles in this flat osculating region become straight lines.

This analogy of the flat football field on the curved surface of the Earth enables us to understand that, particular property of Riemannian geometry which is useful in understanding Einstein's development of the invariant theory of gravitation. The Riemannian geometry on a large portion of the surface of a sphere reduces to Euclidean geometry when the size of the portion of the surface under consideration becomes small.

This is a general property of Riemannian geometry. If in any region of space, the geometry is Riemannian, then in a small neighborhood of a point in the region, the Riemannian geometry reduces to Euclidean geometry. One can express the same property in another way. If in a region of space, the geometry is Riemannian, the region of space can be regarded as curved. At any point in this curved space we can choose a small neighborhood in which the curvature of space is negligible and so the space in this neighborhood can be regarded as flat and Euclidean geometry can be used there.

The Falling Lifts

Let us now return to the falling lift experiment of Einstein. We have found that observer 'A' in the falling lift finds that the path of the chalk-piece is a straight line, while observer 'B' on the ground finds the same path to be parabola. This situation is very much similar to our drawing of straight lines on the football fields and great circles on large portions of the earth; both these being essentially straight lines of Riemannian geometry. There is very simple word viz., the geodesic to denote these straight lines of Riemannian geometry (i.e., shortest path between two points of Riemannian space).

The geodesics on the curved surface of the sphere are great circles. In the football field these geodesics become straight lines. The observations of the falling lift suggest that the path of the chalk-piece must be a geodesic of Riemannian geometry. The falling lift can be likened to a small neighborhood of a point (or to the football field on the earth) and so the geometry in this small region becomes Euclidean and the geodesics become straight lines. For the observer 'B', the curvature is not negligible and he finds the geodesics to be parabolas.

Let us pursue this suggestion a little further. We have mentioned above that the falling lift is a small neighborhood of a point. We should really say, that is a small neighborhood limiting the experiences of the observer 'A'. The smallness of this neighborhood is judged in comparison with the region of experience of the observer B on the ground. Again this smallness does not refer only to the smallness in length, width and the height of the lift (the three dimensions of the lift) but also to the small time-interval to which the experiences of 'A' are limited. The observations of 'A' are not in a small lift but in a small falling lift. The smallness of the region of experience of 'A' is not only in the 3 dimensions of space but also in the fourth 'time-dimension'.

Einstein's Law of Gravitation

We have seen above that the world of experience of an observer is 4-dimensional. The falling lift experiment clearly suggested that gravitation had to be linked up with geometry and that this geometry had to be four dimensional.

Einstein's law of gravitation requires that the geometry governing the observations in a gravitational field must be Riemannian. In other words, the four dimensional region of experience of an observer in a gravitational field is curved, the geometry being Riemannian. Einstein gave a formula to find the curvature of the world of experience, depending on the masses producing the gravitational fields. If one takes a limited region of this world of experience (like the falling lift), the curvature can be neglected and the Riemannian geometry becomes Euclidean.

Thus when the two observers 'A' and 'B' in the thought experiment of Einstein described gravitational phenomena on the basis of Newton's law their descriptions were different. But when they describe the same phenomena on the basis of Einstein's law their descriptions become identical.

Mathematics and Sciences

Einstein's development of the invariant theory of gravitation clearly brings to light the important role which pure mathematics plays in the development of science. Euclid developed an axiomatic system of geometry. Scores of geometers after him attempted to replace some of these axioms and thus produce new geometries. Lobatchewsky and Riemann succeeded in developing two alternative geometries. One may just wonder—why this intense intellectual activity spread over some two thousand years. None of the Mathematicians engaged in this activity had ever suspected that, what they were discovering would someday be used in developing a theory of gravitation. When a Mathematician works out new disciplines in this subject he is working to satisfy his curiosity, his sense of abstraction and generalisation; in short, he is developing mathematics for the sake of mathematics. A scientist on the other hand is interested in observing, studying and understanding nature. In this process a stage comes when he has to look to disciplines developed by pure Mathematicians to gain a better understanding of nature. With this better understanding about nature, the scientist proceeds to develop better methods of harnessing the forces of nature in the service of the society. In this way the contact is finally made with the man in the street.

While the scientist, in this way, goes on converting the mathematical developments into useful tools in the service of society, the pure Mathematician goes on developing more and more pure mathematics tempting future scientists to develop more and more useful tools and in this way the caravan of progress remains continuously on the move.