UNCONVENTIONAL PROBLEMS OF RATIO AND PROPORTION CONCERNING DISTRIBUTION OF THREE LIQUIDS AND THREE ROOMS

S.N. Maitra

Department of Mathematics
National Defence Academy, Khadakwasla
Pune-Thane, Maharashtra
Email: soumen_maitra@yahoo.co.in

Each of the three glass jars contains three different liquids whose volume ratios and density ratios are given. These liquids are poured into and blended in a bigger glass jar. The volume of the blended liquid in the bigger jar is measured. This paper is aimed at finding the mass distribution of the three liquids in the three smaller jars, respectively. Another problem of ratio and proportion is introduced: ratios of height, length and breadth of each of the three rooms and the areas of their floors and walls are given. Aforementioned dimensions of the rooms are calculated.

Keywords: Liquids, Volume, Mass, Blended, Ratio, Proportion, Area, Length, Breadth, Height, Rooms

Introduction

Many problems on ratio and proportion, along with their solutions or without solutions, have appeared in X or XI standard textbooks of mathematics. In this article, two innovative problems involving ratio and proportion are introduced and solved.

Problem 1

This is depicted elaborately as follows:

- Jar 1 contains three different liquids, viz., Liquid 1, Liquid 2 and Liquid 3 in volume ratio $a_{11}:a_{12}:a_{13}$.
- Jar 2 contains the above-mentioned three liquids in volume ratio $a_{21}:a_{22}:a_{23}$.
- Jar 3 contains the above-mentioned three liquids in volume ratio $a_{31}:a_{32}:a_{33}$.

The liquids from these jars are poured into a larger jar and blended. Thereafter, the three different liquids contained in the larger jar are found in volume ratio $a:b:c$.

The total volume and mass of the mixture of three different liquids in the bigger jar are measured to be V and M, respectively.

Additionally, the densities of the three different liquids are given in the ratio $p:q:r$.

The objective is to calculate the mass of each of the three different liquids in the larger jar, as well as in Jar 1, Jar 2 and Jar 3, respectively.

Solution to Problem 1

Let v_{11}, v_{12} and v_{13} be volumes of the three different liquids in Jar 1. Then, as above —

$$\frac{v_{11}}{a_{11}}=\frac{v_{12}}{a_{12}}=\frac{v_{13}}{a_{13}}=\frac{v_{11}+v_{12}+v_{13}}{a_{11}+a_{12}+a_{13}}=k_1\ (\text{say}) \qquad (1)$$

Let v_{21}, v_{22} and v_{23} be the volumes of the three different liquids in Jar 2. Then, as above—

$$\frac{v_{21}}{a_{21}}=\frac{v_{22}}{a_{22}}=\frac{v_{23}}{a_{23}}=\frac{v_{21}+v_{22}+v_{23}}{a_{21}+a_{22}+a_{23}}=k_2\ (\text{say}) \qquad (2)$$

Let v_{31}, v_{32} and v_{33} be the volumes of the three different liquids in Jar 3. Then, as above—

$$\frac{v_{31}}{a_{31}}=\frac{v_{32}}{a_{32}}=\frac{v_{33}}{a_{33}}=\frac{v_{31}+v_{32}+v_{33}}{a_{31}+a_{32}+a_{33}}=k_3\ (\text{say}) \qquad (3)$$

If V_a, V_b and V_c be the volumes of the three different liquids, respectively in the bigger jar, in view of the above.

$$\frac{V_a}{a}=\frac{V_b}{b}=\frac{V_c}{c}=k$$

In consequence of (1), (2), (3) are obtained the volumes of the Liqiud 1, Liquid 2 and Liquid 3 in the bigger jar, respectively as (k_1, k_2, k_3 and k are constant)

$V_a = v_{11} + v_{12} + v_{13} = k_1 a_{11} + k_2 a_{12} + k_3 a_{13} = ka$

$V_b = v_{21} + v_{22} + v_{23} = k_1 a_{21} + k_2 a_{22} + k_3 a_{23} = kb$

$V_c = v_{31} + v_{32} + v_{33} = k_1 a_{31} + k_2 a_{32} + k_3 a_{33} = kc$ (4)

Given that $V_a + V_b + V_c = V$ and as such from (4):

$$k(a + b + c) = V \qquad (5)$$

$$k_1 a_{11} + k_2 a_{12} + k_3 a_{13} = \frac{Va}{a+b+c} \qquad (6)$$

$$k_1 a_{21} + k_2 a_{22} + k_3 a_{23} = \frac{Vb}{a+b+c} \qquad (7)$$

$$k_1 a_{31} + k_2 a_{32} + k_3 a_{33} = \frac{Vc}{a+b+c} \qquad (8)$$

We shall solve the equations (6), (7) and (8) to find the values of k_1, k_2 and k_3 with the help of determinants using Cramer's rule.

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (9)$$

$$k_1=\frac{V}{(a+b+c)D}\begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix}$$

$$k_2=\frac{V}{(a+b+c)D}\begin{vmatrix} a_{11} & a & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & c & a_{33} \end{vmatrix}$$

$$k_3=\frac{V}{(a+b+c)D}\begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix} \qquad (10)$$

Now using equations (1), (2) and (3), we find volumes of the three different liquids in Jar 1, Jar 2 and Jar 3, respectively:

In Jar 1,

$$v_{11}=a_{11}k_1=\frac{Va_{11}}{(a+b+c)D}\begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix} \qquad (11.1)$$

$$v_{12}=a_{12}k_1=\frac{Va_{12}}{(a+b+c)D}\begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix} \qquad (11.2)$$

$$v_{13}=a_{13}k_1=\frac{Va_{13}}{(a+b+c)D}\begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix} \qquad (11.3)$$

In Jar 2,

$$v_{21}=a_{21}k_2=\frac{Va_{21}}{(a+b+c)D}\begin{vmatrix} a_{11} & a & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & c & a_{33} \end{vmatrix} \qquad (12.1)$$

$$v_{22} = a_{22}k_2 = \frac{Va_{22}}{(a+b+c)D}\begin{vmatrix} a_{11} & a & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & c & a_{33} \end{vmatrix} \qquad (12.2)$$

$$v_{23} = a_{23}k_2 = \frac{Va_{23}}{(a+b+c)D}\begin{vmatrix} a_{11} & a & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & c & a_{33} \end{vmatrix} \qquad (12.3)$$

In Jar 3,

$$v_{31} = a_{31}k_3 = \frac{Va_{31}}{(a+b+c)D}\begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix} \qquad (13.1)$$

$$v_{32} = a_{32}k_3 = \frac{Va_{32}}{(a+b+c)D}\begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix} \qquad (13.2)$$

$$v_{33} = a_{33}k_3 = \frac{Va_{33}}{(a+b+c)D}\begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix} \qquad (13.3)$$

If d_a, d_b and d_c be the densities of three different liquids in the bigger jar, then—

$$\frac{d_a}{p} = \frac{d_b}{q} = \frac{d_c}{r} = h \text{ (say)} \qquad (14)$$

Combining (14) with equations (6), (7) and (8), the masses of the three different liquids are obtained as

$$m_a = d_a V_a = hp\,(k_1 a_{11} + k_2 a_{12} + k_3 a_{13})$$

$$= hp\frac{Va}{a+b+c} \qquad (15.1)$$

$$m_b = d_b V_b = hq\,(k_1 a_{21} + k_2 a_{22} + k_3 a_{23})$$

$$= hq\frac{Vb}{a+b+c} \qquad (15.2)$$

$$m_c = d_c V_c = hr\,(k_1 a_{31} + k_2 a_{32} + k_3 a_{33})$$

$$= hr\frac{Vc}{a+b+c} \qquad (15.3)$$

$$M = m_a + m_b + m_c$$

$$= hV\left(p\frac{a}{a+b+c} + q\frac{b}{a+b+c} + r\frac{c}{a+b+c}\right)$$

Or, $$M = \frac{hV}{a+b+c}\,(pa+qb+rc) \qquad (16)$$

which gives the value of h:

$$h = \frac{M(a+b+c)}{V(pa+qb+rc)} \qquad (17)$$

Substituting (17) in (14), one gets the densities of the three different liquids:

$$d_a = \frac{Mp(a+b+c)}{V(pa+qb+rc)}$$

$$d_b = \frac{Mq(a+b+c)}{V(pa+qb+rc)}$$

$$d_c = \frac{Mr(a+b+c)}{V(pa+qb+rc)} \qquad (18)$$

Combining (17) with (15.1), (15.2), (15.3) we get masses of the three different liquids in the bigger jar:

$$m_a = \frac{Map}{(pa+qb+rc)}$$

$$m_b = \frac{Mbq}{(pa+qb+rc)}$$

$$m_c = \frac{Mcr}{(pa+qb+rc)} \qquad (19)$$

Combining (18) with relations (11.1) to (11.3), (12.1) to (12.3), (13.1) to (13.3), the mass distribution of Liquid 1, Liquid 2 and Liquid 3 in Jar 1, Jar 2 and Jar 3, respectively are obtained —

In jar 1,

$$m_{11} = v_{11}d_a = \frac{a_{11}}{D}\frac{Mp}{(pa+qb+rc)}\begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix} \quad (20.1)$$

$$m_{12} = v_{12}d_b = \frac{a_{12}}{D}\frac{Mq}{(pa+qb+rc)}\begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix} \quad (20.2)$$

$$m_{13} = v_{13}d_c = \frac{a_{13}}{D}\frac{Mr}{(pa+qb+rc)}\begin{vmatrix} a & a_{12} & a_{13} \\ b & a_{22} & a_{23} \\ c & a_{32} & a_{33} \end{vmatrix} \quad (20.3)$$

In jar 2,

$$m_{21} = v_{21}d_a = \frac{a_{21}}{D}\frac{Mp}{(pa+qb+rc)}\begin{vmatrix} a_{11} & a & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & c & a_{33} \end{vmatrix} \quad (21.1)$$

$$m_{22} = v_{22}d_b = \frac{a_{22}}{D}\frac{Mq}{(pa+qb+rc)}\begin{vmatrix} a_{11} & a & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & c & a_{33} \end{vmatrix} \quad (21.2)$$

$$m_{23} = v_{23}d_c = \frac{a_{23}}{D}\frac{Mr}{(pa+qb+rc)}\begin{vmatrix} a_{11} & a & a_{13} \\ a_{21} & b & a_{23} \\ a_{31} & c & a_{33} \end{vmatrix} \quad (21.3)$$

In jar 3,

$$m_{31} = v_{31}d_a = \frac{a_{31}}{D}\frac{Mp}{(pa+qb+rc)}\begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix} \quad (22.1)$$

$$m_{32} = v_{32}d_b = \frac{a_{32}}{D}\frac{Mq}{(pa+qb+rc)}\begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix} \quad (22.2)$$

$$m_{33} = v_{33}d_c = \frac{a_{33}}{D}\frac{Mr}{(pa+qb+rc)}\begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix} \quad (22.3)$$

Problem 2

There are three rooms, identified as Room 1, Room 2 and Room 3 of different dimensions that is, length, breadth and height.

Heights of Room 1, Room 2 and Room 3 are in the ratio $h_1: h_2: h_3: : z_1: z_2: z_3$.

Lengths of Room 1, Room 2 and Room 3 are in the ratio $l_1: l_2: l_3 : : y_1: y_2: y_3$.

Breadths of Room 1, Room 2 and Room 3 are in the ratio $b_1: b_2: b_3 : : x_1: x_2 : x_3$.

The surface areas of the walls built up on the lengths, breadths and of the floors of the three rooms are in the ratio $2A_{hl} : 2A_{hb} : 2A_{bl}: : 2a:2b:2c$.

The total surface area of three rooms is measured to be $2S$.

The objective is to find the dimensions of each of the three rooms.

Solution to Problem 2

Owing to the above given data, we have the constants z, y, x:

$$\frac{h_1}{z_1} = \frac{h_2}{z_2} = \frac{h_3}{z_3} = z \quad (1)$$

$$\frac{l_1}{y_1} = \frac{l_2}{y_2} = \frac{l_3}{y_3} = y \quad (2)$$

$$\frac{b_1}{x_1} = \frac{b_2}{x_2} = \frac{b_3}{x_3} = x \quad (3)$$

Multiplying (1) by (2), (2) by (3) and (3) by (1) to obtain the ratio distribution of the surface areas of the rooms:

$$\frac{h_1}{z_1}\frac{l_1}{y_1} = \frac{h_2}{z_2}\frac{l_2}{y_2} = \frac{h_3}{z_3}\frac{l_3}{y_3} = zy$$

$$= \frac{h_1 l_1 + h_2 l_2 + h_3 l_3}{z_1 y_1 + z_2 y_2 + z_3 y_3}$$

$$= \frac{A_{hl}}{z_1 y_1 + z_2 y_2 + z_3 y_3} \qquad (4)$$

$$\frac{b_1}{x_1}\frac{l_1}{y_1} = \frac{b_2}{x_2}\frac{l_2}{y_2} = \frac{b_3}{x_3}\frac{l_3}{y_3} = xy$$

$$= \frac{A_{bl}}{x_1 y_1 + x_2 y_2 + x_3 y_3} \qquad (5)$$

$$\frac{h_1}{z_1}\frac{b_1}{x_1} = \frac{h_2}{z_2}\frac{b_2}{x_2} = \frac{h_3}{z_3}\frac{b_3}{x_3} = zx$$

$$= \frac{A_{hb}}{z_1 x_1 + z_2 x_2 + z_3 x_3} \qquad (6)$$

where 2 A_{hl} and $2A_{hb}$ are areas of the walls built up on the lengths and breadths of the three rooms respectively and $2A_{bl}$ the areas of the floors and roofs.

From the above text, we can write:

$$\frac{2A_{hl}}{2a} = \frac{2A_{bl}}{2b} = \frac{2A_{hb}}{2c}$$

$$= \frac{2A_{hl} + 2A_{bl} + 2A_{hb}}{2(a+b+c)} = \frac{S}{a+b+c} \qquad (7)$$

$$A_{hl} = \frac{Sa}{a+b+c}$$

$$A_{hb} = \frac{Sb}{a+b+c}$$

$$A_{bl} = \frac{Sc}{a+b+c} \qquad (8)$$

Combining equation (8) with (4), (5) and (6), respectively to obtain the relationship:

$$zy = \frac{Sa}{(z_1 y_1 + z_2 y_2 + z_3 y_3)(a+b+c)} = p\,(\text{say}) \qquad (9)$$

$$xy = \frac{Sb}{(x_1 y_1 + x_2 y_2 + x_3 y_3)(a+b+c)} = r\,(\text{say}) \qquad (10)$$

$$zx = \frac{Sc}{(z_1 x_1 + z_2 x_2 + z_3 x_3)(a+b+c)} = q\,(\text{say}) \qquad (11)$$

In order to solve equations (9), (10) and (11), we multiply them to obtain—

$$(xyz)^2 = pqr \quad \text{or,} \quad xyz = \sqrt{pqr} \qquad (12)$$

Dividing (12) by (9), (11) and (10), we get

$$x = \sqrt{\frac{qr}{p}}, \quad y = \sqrt{\frac{pr}{q}}, \quad z = \sqrt{\frac{pq}{r}} \qquad (13)$$

Combining the values of z, y and x from (13) with (1), (2) and (3) to get the dimensions of the three rooms —

$$h_1 = z_1\sqrt{\frac{pq}{r}}, \quad h_2 = z_2\sqrt{\frac{pq}{r}}, \quad h_3 = z_3\sqrt{\frac{pq}{r}} \qquad (14)$$

$$l_1 = y_1\sqrt{\frac{pq}{q}}, \quad l_2 = y_2\sqrt{\frac{pq}{q}}, \quad l_3 = y_3\sqrt{\frac{pq}{q}} \qquad (15)$$

$$b_1 = x_1\sqrt{\frac{qr}{p}}, \quad b_2 = x_2\sqrt{\frac{qr}{p}}, \quad b_3 = x_3\sqrt{\frac{qr}{p}} \qquad (16)$$

By substituting the values of p, q, r from (9), (10), (11) into equations (14), (15) and (16), we obtain –

$$h_1 = z_1\sqrt{\frac{Sac(x_1 y_1 + x_2 y_2 + x_3 y_3)}{(z_1 y_1 + z_2 y_2 + z_3 y_3)(z_1 x_1 + z_2 x_2 + z_3 x_3)(a+b+c)b}} \qquad (17)$$

$$h_2 = z_2\sqrt{\frac{Sac(x_1 y_1 + x_2 y_2 + x_3 y_3)}{(z_1 y_1 + z_2 y_2 + z_3 y_3)(z_1 x_1 + z_2 x_2 + z_3 x_3)(a+b+c)b}} \qquad (18)$$

$$h_3 = z_3\sqrt{\frac{Sac(x_1 y_1 + x_2 y_2 + x_3 y_3)}{(z_1 y_1 + z_2 y_2 + z_3 y_3)(z_1 x_1 + z_2 x_2 + z_3 x_3)(a+b+c)b}} \qquad (19)$$

$$l_1 = y_1\sqrt{\frac{Sba(x_1 z_1 + x_2 z_2 + x_3 z_3)}{(z_1 y_1 + z_2 y_2 + z_3 y_3)(x_1 y_1 + x_2 y_2 + x_3 y_3)(a+b+c)c}} \qquad (20)$$

$$l_2 = y_2\sqrt{\frac{Sba\left(x_1z_1 + x_2z_2 + x_3z_3\right)}{\left(z_1y_1 + z_2y_2 + z_3y_3\right)\left(x_1y_1 + x_2y_2 + x_3y_3\right)\left(a+b+c\right)c}} \quad (21)$$

$$l_3 = y_3\sqrt{\frac{Sba\left(x_1z_1 + x_2z_2 + x_3z_3\right)}{\left(z_1y_1 + z_2y_2 + z_3y_3\right)\left(x_1y_1 + x_2y_2 + x_3y_3\right)\left(a+b+c\right)c}} \quad (22)$$

$$b_1 = x_1\sqrt{\frac{Sbc\left(z_1y_1 + z_2y_2 + z_3y_3\right)}{\left(x_1y_1 + x_2y_2 + x_3y_3\right)\left(z_1x_1 + z_2x_2 + z_3x_3\right)\left(a+b+c\right)a}} \quad (23)$$

$$b_2 = x_2\sqrt{\frac{Sbc\left(z_1y_1 + z_2y_2 + z_3y_3\right)}{\left(x_1y_1 + x_2y_2 + x_3y_3\right)\left(z_1x_1 + z_2x_2 + z_3x_3\right)\left(a+b+c\right)a}} \quad (24)$$

$$b_3 = x_3\sqrt{\frac{Sbc\left(z_1y_1 + z_2y_2 + z_3y_3\right)}{\left(x_1y_1 + x_2y_2 + x_3y_3\right)\left(z_1x_1 + z_2x_2 + z_3x_3\right)\left(a+b+c\right)a}} \quad (25)$$

Thus, equations (17) to (25) represent dimensions of the three rooms evaluated with the knowledge and application of 'Ratio and Proportion'. If we consider each room as a cuboid of unit dimension leading to $x_i = y_i = z_i = 1$, (i = 1, 2, 3), then $a = b = c = 1$, $S = 9$. Substituting these values in the preceding equations are obtained $h_i = l_i = b_i = 1$ (i = 1, 2, 3), which verifies the above results.

References

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