ALGORITHMIC APPROACH IN ANCIENT INDIAN MATHEMATICS AND ASTRONOMY

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Ganita (Indian Mathematics) is the science of computation. Indian Mathematical Texts are compendium of rules for effective methods of computation. These methods are encoded in the form of verses which are predominantly algorithmic in nature. A couple of fascinating algorithms that are representative of Indian mathematical traditions are discussed in this article. Beginning from the algorithm to find the square, square root, cube roots devised by Aryabhata, Bhaskara and Brahmagupta. Then we have a look at algorithm for finding the time from shadow of an instrument called "sanku" and algorithms to find the cardinal directions using the same instrument sanku.

Keywords: Algorithms, Indian Mathematics, Indian Astronomy, *Sanku*, Cube root, Square Root, Cardinal Directions, Time from Shadow

Introduction

Algorithms serve as small procedures that solve a recurrent problem. The word "Algorithm" is derived from the name of Al-Khwarizmi who played a crucial role in the transmission of Indian algorithmic procedures to the Islamic, and later, to the Western world. He is also known as by others "The father of Algebra".

Today we use algorithms like sorting algorithms, Fourier transformation algorithms, encryption algorithms, even face detection algorithms in our everyday life, from making a simple video call to solving complex differential equations. It shouldn't come as a surprise to know that ancient Indian mathematicians and astronomers formulated general algorithms to solve problems. Even other disciplines of Indian tradition are oriented mainly in this fashion where the algorithms for certain *kriyas* (procedures) are presented in the form of *sutras*. In this text, we describe a few representative algorithms in Indian arithmetic as they had evolved over time. 'Aryabhatiya' written in 5th century is one of the earliest texts that contain, algorithms for computing the square root and cube root, etc., of a given number. We then look at the algorithm for calculation of time from shadow measurements and the estimation of East-West line using a device called *sanku*.

1. Algorithm to find the square of a number

In Aryabhatiya (Jha, 1949), the following verse appears as a method for finding the squares

अन्त्यपदस्य वर्गं कृत्वा द्विगुणं तदेवचान्त्यपदम् ।
शेषपदैराहन्यात् उत्सार्योत्सार्य वर्गविधौ ॥

Which translates to

(In order to find the square of a number) The square of the ultimate digit is to be placed over it (the number). The remaining digits are doubled and multiplied by the digit right to it (and are to be placed over them respectively). Then, moving the rest by one place, the cycle is repeated. (At last the sum is calculated.)

Below is a working example to find the square of 125.

1	5	6	2	5	
				25	$5^2 = 25$
		4	20		$2^2 = 4$, 5.2.2 = 20
1	4	10			$1^2 = 1$, 2.2.1 = 4, 5.2.1 = 10
1	2	5			

The speciality of this algorithm is that for a general n-digit number to find its square we would need to calculate $\frac{\{n(n+1)\}}{2}$ different multiplications however in the usual method, we are accustomed to requires n^2 multiplications.

2. The square root algorithm

Aryabhata presents his square root extraction method in the accompanying verse:

भागं हरेत् अवर्गात् नित्यं द्विगुणेन वर्गमूलेन।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम्॥

Which translates to (Bhaskara; Pillai, 1958)

(Having subtracted the greatest possible square from the last odd place and after that having recorded the square root of the number subtracted in the line of the square root), divide the even place (standing on the right) by double the root. At that point, having subtracted the square (of the quotient) from the odd place (remaining on the right), set down the quotient at the succeeding place (i.e., on the right of the number already written in the line of the square root). This gives the square root. (Repeat the process if there are still digits on the right.)

The algorithm presented here essentially consists of three steps:

(1) Starting from the least significant digit, group the digits of the given number into two. Then from the remaining (1 or 2) most significant digit(s), which constitutes Varga-sthana, subtract the square of the maximum number that is possible.

(2) After this, along with the remainder bring down the next digit from the avarga place. This has to be divided by twice the Varga-mula currently in the square root line and the quotient has to be taken to that line.

(3) Along with the remainder, set down the next digit in the Varga place and subtract from it the square of the previous quotient

This procedure is carried to the number 2989441 as an example below and the square root is found systematically to be 1729, the Hardy-Ramanujan number.

		V	*A*	*V*	*A*	*V*	*A*	*V*	1729
		2	9	8	9	4	4	1	(line of square root)
Subtract 1^2		1							
Divide by 2.1	2)	1	9	(7					
		1	4						
Subtract 7^2			5	8					
Divide by 2.17			4	9					
	34)			9	9	(2			
Subtract 2^2				6	8				
Divide by 2.172				3	1	4			
						4			
Subtract 9^2	344)			3	1	0	4	(9	
				3	0	9	6		
							8	1	
							8	1	
								0	

3. Cube root algorithm

The first clear articulation of cube root algorithm is found in Aryabhatiya in the following verse:

अघनात् भजेत् द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण।
वर्गस्त्रिपूर्वगुणितः शोध्यः प्रथमात् घनश्च घनात्॥

Which translates to (Stillwell, 2002; Pillai 1958)

(Having subtracted the greatest possible cube from the last cube place and then having written down the cube root of the number subtracted in the line of the cube root), divide the second non-cube place (standing on the right of the last cube place) by thrice the square of the cube root (already obtained); (then) subtract from the first non-cube place (standing on the right of the second non-cube place) the square of the quotient multiplied by thrice the previous (cube root); and (then subtract) the cube (of the quotient) from the cube place (standing on the right of the first non-cube place) (and write down the quotient on the right of the previous cube root in the line of the cube root, and treat this as the new cube root. Repeat the process if there is still digits on the right).

The algorithm presented by these four steps:

(1) Starting from the unit's place, having grouped the digits of the given number into three, from the remaining (1, 2 or 3) most significant digit(s), which constitutes ghanasthana, subtract the cube of the max. digit that is possible. This digit forms the first (most significant) digit of the cube root to be determined.

(2) Then, along with the remainder, bring down the next digit from the dvitıya-agana place. This has to be divided by thrice the square of the ghana-mula obtained so far. The quotient forms the next digit of the cube root.

(3) Along with the remainder, bring down the next digit in the Prathama-aghana place, and subtract from it the square of the previous quotient multiplied by three and the Purva, the cube root determined previously, that is till now

(4) Finally, from the successive ghana place, we have to subtract the cube of the quotient that was determined previously (from the second step), and the whole process has to be repeated.

We take the cube root of 1773561 as an example to explain the rule (3).

We use c,n,n to represent the cube digit and the first and the second non-cube digits.

		c	n'	n	c	n'	n	c	121
		1	7	7	1	5	6	1	(line of cube root)
Subtract 1^3		1							
Divide by 3.1^2	3)	0	7	(2					
		0	6						
Subtract $3.1.2^2$			1	7					
			1	2					
				5	1				
Subtract 2^3				0	8				
Divide by 3.12^2	432)			4	3	5	(1		
				4	3	2			
Subtract $3.12.1^2$						3	6		
						3	6		
Subtract 1^3							0	1	
								1	
								0	

4. Algorithm to accurately determine cardinal directions

Undeniably, a lot of mathematics and art of measurement goes hand-in-hand with astronomy. Knowledge of accurate direction of East is very important for the performance of *yagna* and other religious performances.

The procedure to find the East-West line with the help of a device called *sanku* is described in the following verse: Parakh, 2006.

समे शङ्कुं निखाय 'शङ्कुसम्मितया रज्ज्वा' मण्डलं परिलिख्य यत्र लेखयोः शङ्कुग्रच्छाया निपतति तत्र शङ्कू निहन्ति, सा प्राची।

This can be interpreted as

Fixing a sanku on levelled ground and drawing a circle with a cord measurable by the sanku (implying the circle's radius to be greater than twice the *sanku* height), he fixes pins at points on the line (of the circumference) where the shadow of the tip of the sanku falls. That gives the east-west line.

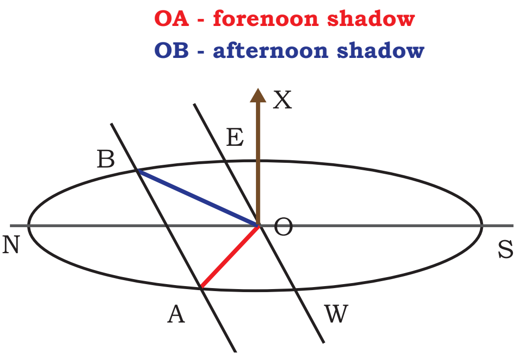

Fig. 1. Diagram showing the use of sanku to determine the East-West line

The device called *sanku* is made up of a rod of 12 angula height. It is depicted as OX in the image (Fig.3). It is equivalent to the modern projecting piece found on a sundial. Now one would think that it is easier to just look at the direction of sunrise to determine the East direction, why bother with performing such an activity?

The following verse answers that question:

...तस्य उदयस्थानानां बहुत्वात् प्रतिदिनं भिन्नत्वात् अनियमेन प्राची ज्ञातुं न शक्या। तस्मात् शङ्कुस्थापनेन प्राचीसाधनमुक्तम्। दक्षिणायने चित्रापर्यन्तमर्कोऽभ्युदेति। मेषतुलासङ्क्रात्यहे प्राच्यां शुद्धायामुदेति। ततोऽर्कात् प्राचीज्ञानं दुर्घटम्।

Which means that

Each day the rising and setting points change slightly. At the summer solstice, the Sun rises as far to the northeast as it ever does and sets as far to the northwest. Every day after that, the Sun rises a tiny bit further south Pillai, 1958.

Since the rising points are many, varying from day-to-day, the [cardinal] east point cannot be known [from the sunrise point]. Therefore, it has been prescribed that the east be determined by fixing a *sanku*... Therefore, simply looking at the sun and determining the east is difficult.

After establishing the East-West line, we have to determine the North-South line as well which is perpendicular to the East-West line. The following *shloka* explains the procedure to find the perpendicular Pillai, 1958.

तदन्तरं रज्वाभ्यस्य, पाशौ कृत्वा, शङ्कोः पाशौ प्रतिमुच्य, दक्षिणायम्य मध्ये शङ्कुं निहन्ति। एवमुत्तरतः, सोदीची।

Doubling the cord by a measure of distance between them (sanku)...and stretching (the cord) towards the south, strikes a pin at the middle point. The line joining endpoints give the perpendicular (North-South line) (As shown in Fig.2).

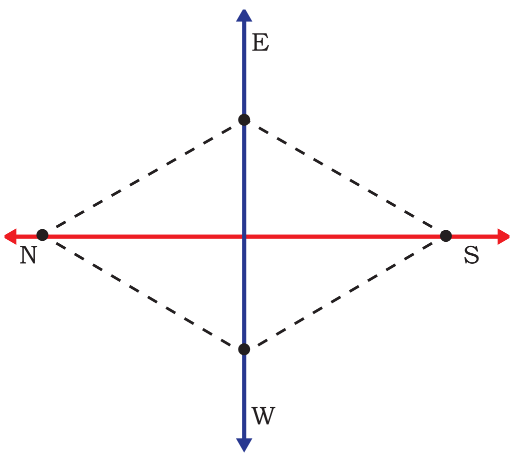

Fig. 2. Figure showing construction of the perpendicular to the East-West line.

5. Algorithm for finding the time from shadow

Estimating time from the shadow of an object is an immensely important especially in the field of astronomy and while navigating the seas.

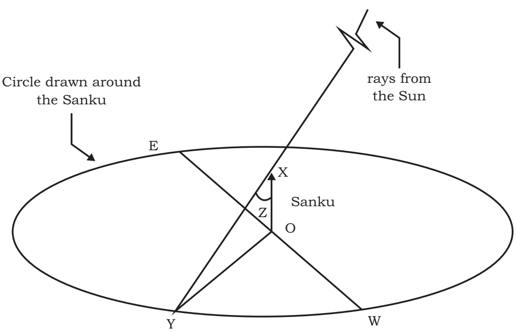

Fig. 3. Diagram showing the use of sanku to calculate the time from the shadow of sanku

Here in Fig. 3, OY represents the shadow casted by sanku (OX).

If the longitude of the sun, λ is known at a given instant, then its declination, δ at that instant can be calculated using the relation $\sin\delta = \sin\varepsilon \sin\gamma$

The method for getting precise estimation of the latitude γ is talked about in great detail in several astronomical texts see for instance (Katyana; NPTEL n.d.)

Nilakantha mentions the algorithm to measure time using these data in the following verse (Ramasubramanian and Sarma 1976).

व्यासार्धघ्नात् ततः शङ्कोः लम्बकाप्तं त्रिजीवया ॥
हत्वा द्युज्याविभक्ते तत् चरज्या स्वर्णमेव च ।
याम्योदग्गोलयोस्तस्य चापे व्यस्तं चरासवः ॥
संस्कार्या गतगम्यास्ते पूर्वापरकपालयोः ।

Which translates simply as

The sanku is multiplied by R and divided by γ. This is further multiplied by *trijya* and divided by *dyujya*. To this quantity *carajya* is applied based on the location.

To the arc of the result, the carasavah has to be applied in the reverse order to get the time elapsed in the eastern hemisphere.

This procedure corresponds to the mathematical equation:

$$t = (R \sin)^{-1}\left[\frac{R \cos z}{\cos\phi \cos\delta} \pm R \sin \Delta\alpha\right] \mp \Delta\alpha$$

Conclusion

There are several hundreds of algorithms developed by ancient Indian mathematicians such as Aryabhata, Bhaskara, Nilakantha, Brahmagupta and others on various topic. Apart from the once presented here, there are the *sulabha sutras*, more rigorous algorithm to find roots, algorithms to find sine table, accurate computation of Π, algorithm to find instantaneous velocity of a planet, to calculate *lagna*, solar distance, etc., among many others. Some of these algorithms are so efficient that they are implemented even today in computation systems. The techniques instructed presently in schools for root extraction are basically the same as those deduced by Aryabhatta hundreds of years back. It is evident from this discussion that Indian texts are formulated with information in the form of algorithms and puzzles rather than bland equations and propositions.

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