# PROBLEM BASED LEARNING IN BASIC PHYSICS – XII

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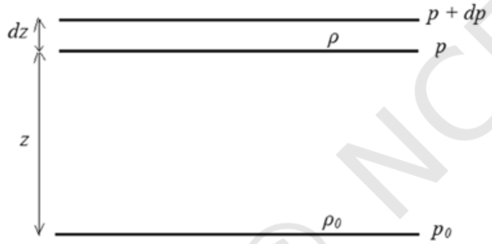
In this article, twelfth in the series, we present two interesting problems for a problem based learning course— one on modelling earth's atmosphere and second-on modelling of Sun based on basic physics. We present the learning objectives in this area of basic physics and what each problem tries to achieve with its solution. Methodology and philosophy of selecting these problems are already discussed.

## Understanding the Earth's Atmosphere




1. Consider an isothermal model of the atmosphere, i.e., there is no change in atmospheric temperature with height.

Let ground values of pressure and density be $p_0$ and $\rho_0$ respectively.

Show that pressure of the earth's atmosphere varies as, where atmospheric pressure is p, and atmospheric density is $\rho$ at a height z. Here, $z_0 = \frac{p_0 g}{\rho_0}$ is called scale height.

Taking $p_0 = 1.013\times10^5$ N/m$^2$ and $\rho_0 = 1.29$ kg/m$^3$, estimate the scale height value. Assuming $e^{-5}\approx0$, estimate thickness of earth's atmosphere for such an isothermal model. Also estimate the pressure at the top of Mt. Everest which has a peak at about 8 km.

2. The following table shows temperature v/s altitude data that was displayed in an Air India flight from Mumbai to New Delhi on 10th August 2014 at around 6:30 pm recorded before landing at New Delhi.

| Height in Metre | Temp in Celsius |
|---|---|
| 300 | 30 |
| 800 | 27 |
| 900 | 26 |
| 1200 | 24 |
| 2100 | 20 |
| 3700 | 11 |
| 5900 | –4 |
| 6300 | –6 |
| 6900 | –10 |
| 7500 | –13 |
| 8100 | –17 |
| 8800 | –22 |
| 9300 | –25 |

| | |
|---|---|
| 10100 | –30 |
| 11300 | –42 |

Clearly this shows that earth's atmospheric temperature does change with height. Check if this change is significant at a magnitude of scale height.

3. Actually air is a bad conductor of heat (to our advantage) and an envelope of air rising does not exchange heat with the surrounding but cools due to adiabatic expansion as a result of drop in the pressure. Refer to the graph in Problem 2 above.

We need to find an expression for *p* as a function of altitude *z* from the fact that experimentally we get $T = -0.0062\,z + 305.582$ (in Kelvin).

**Solution**

1. Starting with ground values of pressure and density be $p_0$ and $\rho_0$ respectively, at a further height pressure will change (decrease) by $dp = -\rho g dz$ at a further height *dz.*

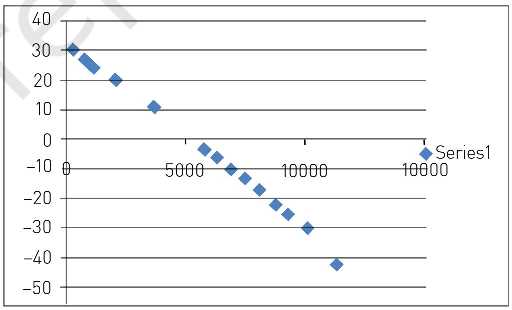
In such an isothermal atmosphere, $p \,\alpha\, \rho$ and hence we can write $\frac{p}{p_0} = \frac{\rho}{\rho_0}$ .

This leads to $dp = -\frac{\rho_0}{p_0} g dz$ , integration of which yields, $p = p_0 \exp(-z/z_0)$ where $z_0 = \frac{p_0 g}{\rho_0}$ is called isothermal scale height of the atmosphere.

At sea level T = 300 K, $\rho = 1.29$ kg/m$^3$, assuming g = 9.81 m/s$^2$ (constant) and universal gas constant R = 8.314 J/mole/K we get $z_0 = 8.58$ km.

According to isothermal condition, atmospheric pressure decreases exponentially with increasing height. Since the temperature is assumed to be constant, and $\rho \propto \frac{p}{T}$, it follows that the density also decreases exponentially with the same scale-height as the pressure. According to the equation above, the pressure, or density, decreases by a factor 10 every 19.8 kilometre, as we move vertically upwards. This indicates that the effective height of the atmosphere is very small compared to the earth's radius, which is about 6400 kilometre. This means that the atmosphere constitutes a very thin layer covering around the surface of the earth. This also justifies our neglect of the decrease of g with increasing altitude.

2. Plot the graph of temperature v/s altitude and find the equation that represents the variation. You may easily do this in MS-EXCEL. The graph below shows plot of the given data with height above earth's surface on x-axis and corresponding temperature on y-axis.




[We get the equation $T = -0.0062\,z + 32.582$, where *z* is in metres.].

3. Result of problem 2, clearly shows that over distance (scale height) $z_0$ (which we found to be 8.85 km), atmospheric temperature does not remain constant and our assumption of isothermal condition is not correct.

We have $\frac{dp}{dz}=-\rho g$ and $pV=nRT$, and $\rho=\frac{nM_A}{V}$ where $M_A$ is the molecular weight of the air, we get $\rho=\frac{pM_A}{RT}$. With $T = T_0 - \alpha z$, we get $\frac{dp}{dz}=-\frac{pM_A g}{R(T_0-\alpha z)}$

Integration yields, $p=p_0\left(1-\frac{\alpha}{T_0}z\right)^{M_A g/RT}=p_0\left(1-\frac{\alpha}{T_0}z\right)^{5.425}$ and we have $\frac{\alpha}{T_0}=2.029\times10^{-5}$. With this pressure drops by factor of 10 every 32.2 km.

In adiabatic approximation,

$p^{1-\gamma}T^{\gamma}$=constant

giving $\frac{dp}{p}=\frac{\gamma}{1-\gamma}\frac{dT}{T}$

Combining the above relation with the equation of hydrostatic equilibrium, we obtain

$$\frac{\gamma}{\gamma-1}\frac{dT}{T}=-\frac{\mu g}{RT}dz\ ,$$

or

$$\frac{dT}{dz}=-\frac{\gamma-1}{\gamma}\frac{\mu}{RT}$$

Since $\rho T \alpha p$ , according to the ideal gas law, then

$$(\rho T)_{packet}=(\rho T)_{atmosphere}$$

This yields $T=T_0\left(1-\frac{\gamma-1}{\gamma}\frac{z}{z_0}\right)$,

and $p=p_0\left(1-\frac{\gamma-1}{\gamma}\frac{z}{z_0}\right)^{\gamma/\gamma-1}$,where $z_0=\frac{RT_0}{\mu g}$

The expression obtained above in the limit of $\gamma\rightarrow1$ will get $p=p_0\exp\left(-\frac{z}{z_0}\right)$.

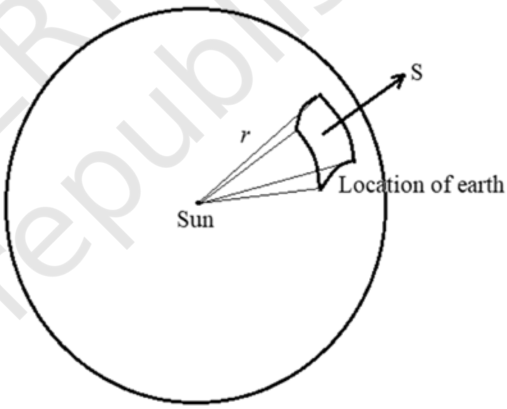
This, not surprisingly, is the predicted pressure variation in an isothermal atmosphere. In reality, the ratio of specific heats of the atmosphere is not unity, it is about 1.4 (i.e., the ratio for diatomic gases), which implies that in the real atmosphere

$$p=p_0\left(1-\frac{z}{3.5z_0}\right)^{3.5}$$

## Model of Sun

In this problem we shall learn to estimate solar luminosity, surface temperature, core temperature, possible mechanism of energy generation inside sun (or stars) and criteria for an astronomical object to become a star.

Energy for survival of life comes from our sun. At the location of earth, we receive $S$ = 1400 W/m$^2$ (called solar constant). If we use this fact and the fact that average distance between earth and sun is 1 astronomical unit (1 AU = $1.5\times10^{11}$m), then referring to the figure below, energy radiated by sun must be L_S=$4\pi r^2$ S=$3.96\times10^{26}$ W. This quantity is called as **solar luminosity**.



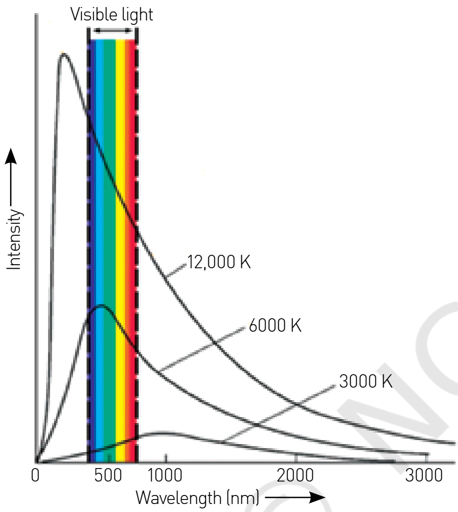

## Surface Temperature of Sun

According to Stefan's Law power radiated by a black body per unit surface area having surface temperature $T_S$ is given by $P=\sigma T_S^4$ , where $\sigma=5.67\times10^{-8}$ W/m$^2$. If radius of sun is $R_S$ , then this gives us $L_s=4\pi R_S^2\sigma T_S^4$ which gives

$$T_s=\left[\frac{L_S}{4\pi R_S^2\sigma}\right]^{1/4}=5781\text{ K}$$

as the surface temperature of the sun. Where we have used radius of the sun as $R_S$=$7\times10^8$m. Many times this is also referred to as black body temperature of solar surface as we have approximated sun's surface here as a black body.

Another way of estimating surface temperature of the sun is using Wien's Law. The energy radiated by black body is radiated at different wavelengths at different temperatures is given by graphs shown in the following figure.




According to Wien's Law, if a black body radiates maximum energy at wavelength $\lambda_m$, then it has surface temperature $T_S$, given by

$$T_s = \left[\frac{0.29\text{cmK}}{\lambda_m}\right].$$

Sun radiates maximum energy in the visible region of spectrum, estimated to be 500 nm. This gives us its surface temperature as 5800 K. This is almost same as our previous estimate.

### The Process of Energy Generation Inside the Sun

The energy radiated from the surface must come from its interior which must be hotter than the surface. What must be the mechanism for generation of this amount of energy?

Three possibilities are:

(i) Gravitational potential energy

(ii) Chemical energy due to chemical reactions in the interior

(iii) Nuclear energy due to nuclear processes

Let us examine the consequence of each possiblility:

### Gravitational Potential Energy

Sun may have been a big gas cloud which due to gravitational force has shrunk to its present size.

Gravitational potential energy of a spherical object of mass M and radius R is

$$U_G = -\frac{3}{5}\frac{GM^2}{R}$$

This energy is zero as $R \to \infty$ but for present radius is

$$U_G = -2.286\times10^{41}\text{J}\ .$$

This means that sun became hot as it contracted and heat energy came from gravitational potential energy.

With this energy and current luminosity, sun can continue shining for time of

$$\Delta t = \frac{U_G}{L_S} = 5.7\times10^{14}\text{S} = 1.8\times10^{7}\ \text{years}$$

From radioactive dating, earth's present age is estimated to be $5\times10^9$ years. Clearly, gravitational energy could not be the source of sun's luminosity.

### Chemical Energy

Typical energy released during a chemical reaction is of the order of an electron volt.

If all of sun is H-atoms, then number of H-atoms : $n = \frac{M_S}{m_H} = 1.2\times10^{57}$

If two $^1H$ combine to form $H_2$, the number of reactions $= \frac{n}{2} = 6\times10^{56}$

This can produce total energy

$U_{chemical} = 6\times10^{56}\times1eV = 9.6\times10^{37}\,J$

$\Delta t = \frac{U_{chemical}}{L_S} = 2.4\times10^{11}S = 7674$ years

This is too small. Thus, chemical reactions cannot be source of Solar Energy.

### Nuclear Energy

There are two competing processes by which $4\,^1H \rightarrow {}^4He$ releasing 26.7 MeV

For This $3\times10^{56}$ reactions would produce 1.28×1045 *J*

$\Delta t = \frac{U_{nuclear}}{L_S} = 3.24\times10^{18}S = 10\times10^{10}$ years

This appears to be the main source of Solar Energy.

### How hot should the core be to initiate nuclear fusion reaction?

To trigger nuclear reaction : to fuse two protons, we need to overcome electrostatic repulsion between two protons.

This will be possible only if the gas is vey HOT. If we assume that initially when the cloud collapsed, its gravitational energy will get converted to thermal energy then gas gets heated up.

If $\Delta U_{Gravity} = \frac{3}{2}nk_BT_c$ then we get

$$T_c = \frac{\Delta U_{Gravity}}{\frac{3}{2}nk_B} = \frac{2.286\times10^{41}}{\frac{3}{2}\times1.2\times10^{57}\times1.38\times10^{-23}} = 9.2\times10^6K$$

Will this temperature be sufficient to ignite nuclear reaction?

If at the time when gas cloud contracted, if all of it is H-atoms, then what will happen?

H-atom has BE of 13.6 eV. If this energy is of the order of $k_BT$, then equivalent temperature is of the order of $10^5$ K. Which means at the temperature which star reached due to gravitational collapse, we will have ionized gas, i.e., we will have protons and electrons separated.

### A classical requirement of the temperature at the center of the stars

Assume that the gas that forms the star is pure ionized hydrogen (electrons and protons in equal amounts), and that it behaves like an ideal gas. From the classical point of view, to fuse two protons, they need to get as close as $10^{-15}$ m for the short range strong nuclear force, which is attractive, to become dominant. However, to bring them together they have to overcome first the repulsive action of Coulomb's force. Assume classically that the two protons (taken to be point sources) are moving in an antiparallel way, each with velocity $v_{rms}$, the root-mean-square (rms) velocity of the protons, in a one-dimensional frontal collision.

What has to be the temperature of the gas, $T_c$, so that the distance of closest approach of the protons, $d_c$, equals $10^{-15}$ m?

We equate the initial kinetic energy of the two protons to the electric potential energy at the distance of closest approach:

$$2\left(\frac{1}{2}m_pv_{rms}^2\right) = \frac{q^2}{4\pi\varepsilon_0 d_c}$$

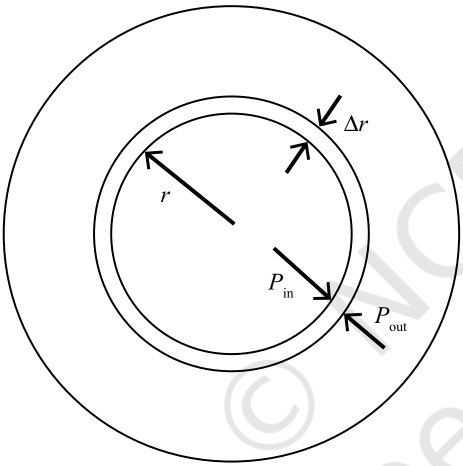
And since $\frac{3}{2}k_BT_C = \frac{1}{2}m_pvp_{rms}^2$

This means we require $T_c = \frac{q^2}{12\pi\varepsilon_0 d_c k_B} = 5.5\times10^9K$

### Two temperatures do not match, then how is nuclear reaction triggered?

To check if the previous temperature estimate is reasonable, one needs an independent way of estimating the central temperature of a star.

The structure of the stars is very complicated, but we can gain significant understanding making some assumptions. Stars are in equilibrium, that is, they do not expand or contract because the inward force of gravity is balanced by the outward force of pressure .




$$\Delta P = P_{in} - P_{out}$$

**Fig.1.** The stars are in hydrostatic equilibrium, with the pressure difference balancing gravity.

Refer to the the figure shown. For a slab of gas the equation of hydrostatic equilibrium at a given distance r from the center of the star, is given by

$$\frac{\Delta P}{\Delta r} = -\frac{GM_r \rho r}{r^2}$$

where P is the pressure of the gas, G the gravitational constant, $M_r$ the mass of the star within a sphere of radius r, and $\rho_r$ is the density of the gas in the slab.

An order of magnitude estimate of the central temperature of the star can be obtained with values of the parameters at the center and at the surface of the star, making the following approximations:

$$\Delta P \approx P_0 - P_C,$$

where $P_C$ and $P_0$ are the pressures at the center and surface of the star, respectively.

Since $P_C >> P_0$, we can assume that $\Delta P = -P_C$

Within the same approximation, we can write $\Delta r \approx R$,

where R is the total radius of the star, and $M_r \approx M_R = M$, with M the total mass of the star.

The density may be approximated by its value at the center, $\rho_r \approx \rho_c$.

We can assume that the pressure is that of an ideal gas.

Thus, to find an equation for the temperature at the center of the star, Tc , in terms of the radius and mass of the star and of physical constants only:

Since we have that $\frac{\Delta P}{\Delta r} = -\frac{GM_r p_r}{r^2}$,

making the assumptions given above, we obtain that: $P_C = \frac{GM \rho c}{R}$,

Now, the pressure of an ideal gas is

$$P_C = \frac{2\rho_c k_B T_c}{mp},$$

where $k_B$ is Boltzmann´s constant, $T_c$ is the central temperature of the star, and $m_p$ is the proton mass. The factor of 2 in the previous equation appears because we have two particles (one proton and one electron) per proton mass and that both contribute equally to the pressure.

Equating the two previous equations, we finally obtain that: $T_C = \dfrac{GMm_p}{2k_BR}$

For the Sun we have that:

$$\frac{M_S}{R_S} = 2.9\times10^{21}\text{kg}/\text{m}$$

$$\therefore T_C = \frac{GM_S m_p}{2k_B R_S} = 11.5\times10^6\text{K}$$

which seems to be close to the temperature of gravitationally collapsed/contracted star.

Our simple estimate requires temperature of $5.5\times10^9$ K to initiate fusion reaction. This criterion we need to reconsider.

**A quantum mechanical estimate of the temperature at the center of the stars**

The large discrepancy found, suggests that the classical estimate for $T_c$ obtained in is not correct. The solution to this discrepancy is found when we consider quantum mechanical effects, that tell us that the protons behave as waves and that a single proton is smeared on a size of the order of , the de Broglie wavelength. This implies that if $d_c$, the distance of closest approach of the protons is of the order of $\lambda_p$, the protons in a quantum mechanical sense overlap and can fuse.

Assuming that dc = $\lambda/2^{1/2}$ is the condition that allows fusion, for a proton with velocity $v_{rms}$, we have $\lambda_p = \dfrac{h}{m_p v_{rms}}, \dfrac{3}{2}k_B T_C = \dfrac{1}{2}m_p v_{rms}^2$ and $T_c = \dfrac{q^2}{2\pi\varepsilon_0 d_c k_B}$

We obtain $T_C = \dfrac{q^4 m_p}{24\pi^2\varepsilon_0^2 k_B h^2} = 9.7\times10^6\text{K}$

This gives $\dfrac{M}{R} = \dfrac{2k_B T_C}{Gm_p} = 2.4\times10^{21}\text{kg}/\text{m}$

While for the Sun we have that:

$$\frac{M_S}{R_S} = 2.9\times10^{21}\text{kg}/\text{m}$$

Thus quantum mechanically temperature required to initiate fusion reaction is same as the temperature available due to gravitational contraction and nuclear process for energy generation through fusion of hydrogen can start.

We now understand what makes Sun shine and generate this tremendous energy.

Considering Solar luminosity and energy per reaction, Sun requires $\dfrac{L_S}{E_{fusion}} = \dfrac{3.96\times10^{26}}{26.7\times1.6\times10^{-13}} = 9.27\times10^{37}$ reactions per second, which means $3.7\times10^{38}$ atoms react per second. Sun at formation had $\dfrac{M_s}{m_p} = \dfrac{2\times10^{30}}{1.67\times10^{-27}} = 1.2\times10^{57}$ H-atoms (or proton). In case of Sun, once core mass of $0.1M_S$ of H is exhausted, there is no more energy production at the centre and it is unable to prevent a small collapse as radiation pressure drops. This heats up the core further and He burning begins. Before discussing He burning, let us estimate the time for core H-burning. Since there are $1.2\times10^{56}$ protons at the core, if Sun continues to shine with same luminosity, it can continue to do so up to $\dfrac{1.2\times10^{56}}{3.7\times10^{38}}$

$=3.2\times10^{17}$ *sec* = $10\times10^{19}$ *years* = 10 billion years. From earlier mentioned age of earth, it appears that Sun is mid way in its H-burning stage.

**Fusing helium nuclei in older stars**

As stars get older they will have fused most of the hydrogen in their cores into helium (He), so they are forced to start fusing helium into heavier elements in order to continue shining. A helium nucleus has two protons and two neutrons, so it has twice the charge and approximately four times the mass of a proton. We saw before that $d_c = \lambda_p / \sqrt{2}$ is the condition for the protons to fuse.

Set the equivalent condition for helium nuclei and find $v_{rms}$ (He), the rms velocity of the helium nuclei and $T$ (He), the temperature needed for helium fusion.

For helium we have :

$$\frac{4q^2}{4\pi\varepsilon_0 mHe v_{rms}^2(He)} = \frac{h}{\sqrt{2}mHe v_{rms}(He)}$$

From where we get $v_{rms}$ (He) $= \frac{\sqrt{2}q^2}{\pi\varepsilon_0 h} = 2.0\times10^6 m/s$

We now use: T(He)= $= \frac{v_{rms}^2(He)m_{He}}{3k_B} = 6.5\times10^8 K$

These values are of the order of magnitude of the estimates of stellar models.

### The mass to radius ratio of the stars

The previous agreement suggests that the quantum mechanical approach for estimating the temperature at the center of the Sun is correct.

We use the previous results to demonstrate that for any star fusing hydrogen, the ratio of mass M to radius R is the same and depends only on physical constants. Let us find the equation for the ratio M / R for stars fusing hydrogen.

Taking into account that $\frac{M}{R} = \frac{2k_BT_c}{Gm_p}$ and that

$$T_c = \frac{q^4 m_p}{24\pi^2\varepsilon_0^2 k_B h^2},$$

We obtain $\frac{M}{R} = \frac{q^4}{12\pi^2\varepsilon_0^2 G h^2}$ the average electron number density inside the star $n_e = \frac{M}{\left(4/3\right)\pi R^3 m_p}$

the typical separation between electrons inside the star $d_e = n_e^{-1/3} = \left[\frac{M}{\left(4/3\right)\pi R^3 m_p}\right]^{-1/3}$

We assume that $d_e \geq \gamma_e / \sqrt{2}$ and since

$$\lambda_e = \frac{h}{m_e v_{rms}(electron)},$$

we get, $\frac{3}{2}k_B T_C = \frac{1}{2}m_e v_{rms}^2(electron)$

where $T_C = \frac{q^4 m_p}{24\pi^2\varepsilon_0^2 k_B h^2}, \frac{M}{R} = \frac{q^4}{12\pi^2\varepsilon_0^2 G h^2}$ and

$$d_e = \left[\frac{M}{\left(4/3\right)\pi R^3 m_p}\right]^{-1/3},$$

we get $R \geq \frac{\varepsilon_0^{1/2} h^2}{4^{1/4} q m_e^{3/4} m_p^{5/4} G^{1/2}} = 6.9\times10^7 m = 0.1R\,(Sun)$

Since $\frac{M}{R} =$ *constant*, it also means

$M \leq M$ (Sun), for a gas cloud to shrink and initiate nuclear fusion to shine and become a star.

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