'BACK-OF-THE-ENVELOPE' ESTIMATION: AN EXCELLENT PEDAGOGICAL TOOL

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Abstract

In this article we point out the merits of back-of-the-envelope estimation as a pedagogical tool, with a few wonderful examples from the history of science and some simple worked out examples from school level science. The same may be utilised by teachers and students to make teaching-learning more engaging, interesting and effective.

What is 'back-of-the-envelope' estimation?

'Back-of-the-envelope' estimation is often used as a tool to gain an insight to a problem or situation, or even to make a prediction through a quick calculation. It literally means a calculation that can be worked out in the small space available on the backside of a conventional postal envelope. In a general sense, it means a calculation that can be written down on any scrap of paper, or even worked out mentally. It usually involves reasonable guesses, plausible approximations or rounding ups and is quite useful for order-of-magnitude estimation. Hence it is also called 'guesstimation' (guess and estimation) or 'order-of-magnitude' estimation. For example, though students are advised to take $9.8 m/s^2$ as the standard value of the acceleration due to gravity (g) for 'exact' calculations, the amount $10\ m/s^2$ is a good enough order-of-magnitude value of g and is used often. While estimating, one should know how to retain what is important and throw away what is not.

Pedagogical significance of back-of-the-envelope estimation

Back-of-the-envelope estimation ought to be an essential component of teaching-learning process as it has great pedagogical virtues. It enables the practitioner. For example,

- to enhance numerical ability and thinking skill,
- to judge the relative importance of the quantities involved,
- to make plausible guesses and approximations,

- to decide when and how to round up numbers,
- to link concepts and principles from different areas of study,
- to broaden perspective,
- to make and verify predictions,
- to make classroom processes exciting and rewarding,
- to use it as a guide for more elaborate or more exact calculation,
- to use it as a study guide,
- to use it as an assessment tool, etc.

Superficially, one may deem back-of-the-envelope estimation as important for students aspiring to become scientists, engineers, accountants, actuaries, business planners, etc. who are supposed to deal with numbers. However, the fact is that almost all of us use rough or approximate calculations even in our everyday life. In view of this, the National Curriculum Framework 2005 (NCF 2005) underlines the importance of approximation and estimation as an essential mathematical skill and advocates its teaching-learning right from primary stage of schooling in the subject domains of mathematics (NCF 2005, p. 43) and science (NCF 2005, p. 48). Needless to say, scientists use it routinely in their own work as well as in teaching their students and research scholars.

The Plan

In this article we shall reproduce or workout a few simple examples of back-of-the-envelope estimation based on school level science and point out their pedagogical features. Before that, as a motivating factor, it may be worthwhile to recount a few famous instances from the history of science (https://en.wikipedia.org/wiki/Back-of-the-envelop_calculation and references therein) to highlight the power of the practice much valued by scientists.

Examples from the History of Science

In 1945, while witnessing the first atomic bomb explosion, Enrico Fermi (who played a very significant role in the science of nuclear fission, one outcome of which was the atomic bomb) dropped a few bits of paper, measured how far they were blown away by the blast wave and then quickly estimated the energy yield of the bomb to be equivalent to 10 kilotons of TNT. This came out to be close to the measured yield of 18.6 kilotons, obtained later.

Though the credit for invention of RADAR (1935) goes to R. A. Watson-Watt, the idea was actually born out of a back-of-the-envelope calculation made by one of his associates, A. F. Wilkins on the use of radio waves in warfare.

C. H. Townes apparently made a few quick calculations on the back of an envelope pulled from his pocket while relaxing in a park and the calculations led to the concepts of MASER and LASER for which he was honoured with the 1964 Nobel Prize in Physics along with two Russian scientists.

In 1989, during a lunch break, a couple of engineers, deeply disturbed by the exploding size of the internet on which they were working, scribbled a new internet protocol on the back of three ketchup-stained napkins; the protocol became famous as the Border Gateway Protocol or the 'three-napkin protocol' and has proved to be highly successful in the management of worldwide internet service.

There are highly successful scientist-teachers who encourage their students to practise back-of-the-envelope calculation by posing interesting questions and themselves working out some of them using this technique (e.g. Weisskoff, 1970; Wikipedia.html - Fermi problem; Wikipedia.html - Fermi questions). An Indian scientist and popular science writer, G. Venkataraman (Venkataraman, 2013) has collected some examples of such simple calculations in his exciting book titled *Why are Things the Way they Are*? We work out below a few examples of plausible guesstimation mostly following the references cited above, for the benefit of students and teachers.

Some Worked out Examples

1. The Fermi Poser

Enrico Fermi, referred to above, is credited with popularisation of back-of-the-envelope calculation in natural sciences to make quick order-of-magnitude estimations. He himself made use of such estimations on several occasions as has been illustrated above. He was also regarded as one of the most popular teachers of his time who used back-of-the-envelope calculation to teach his students. Fermi won the 1938 Physics Nobel Prize for his contribution in nuclear science. As narrated by Venkataraman in his book referred to above, Fermi apparently once asked his students to estimate the number of pianos in the city of Chicago. This is an excellent example of guesstimation which may be carried out in the following manner.

- Let the number of pianos in Chicago be N. The estimation of N may follow the below-mentioned steps, as quoted in Venkataraman, 2013.
- N cannot certainly be greater than the population of the city. Assuming that every person may have one piano at the most and the population of Chicago is one million (10^6) we have $N \leq 10^6$.
- But actually, it is way too much to say that every person in the city has one piano. It is more reasonable to suppose that every family may have one piano each. So, if there are 5 members in each family on the average, then $N \leq (10^6/5 =)$ 200,000.
- Even this is too big a number. All families certainly cannot afford to have piano, a class of costly musical instruments. Let us say only the very affluent can afford it and this is only 20% or 1/5th of the families. Then, $N \approx$ (200,000/5 =) 40,000.
- But now every rich family may not prefer a piano; may be only 30 per cent or 3 out of 10 choose to do so. This then leads to the figure $N \approx$ (40,000 x 3/10=) 12,000, which could be taken as a fairly accurate estimate of the number of pianos in Chicago.

One may not agree to some of the percentages assumed here. But this example illustrates how one makes arguments, guesses, puts upper limits, lower limits, etc. However, to do all these, one needs conceptual clarity and depth.

Many more of such back-of-the-envelope estimates, popularly known as 'Fermi questions' (Wikipedia.html - Fermi questions), are ascribed to Fermi, which include finding the number of piano tuners in a city, number of jelly beans that fill a one-litre bottle, etc.

Recognising that the above examples do not refer to problems in science, let us now see, with the help of a few examples, how back-of-the-envelope calculations can be made to answer some tricky scientific questions without much sweat. For this, we shall use Venkataraman's book referred to above and references therein.

2. How many Types of Atoms one there in Nature?

The conception of atom as the building block of matter, enunciated by Indian philosophers during 800 BC and by the Greek philosopher Democritus in 400 BC, was definitely the manifestation of genius of the highest order. Then, after a long gap, came Dalton's atomic theory in the first decade of 19th century. Subsequently, scientists discovered that there were just 92 naturally occurring elements. Of course, new elements are being made in the laboratory and added to the periodic table off and on, the latest number of elements being 118. Most of these elements are short lived and do not occur naturally.

Why only 92 elements occur naturally, or why the atomic number Z is restricted to 92 in case of naturally occurring elements is a worthwhile question to ponder.

As quoted by Venkataraman in his book, S. Chandrasekhar (Physics Nobel Laureate of 1983) tried to give an answer using a back-of-the-envelope calculation. It uses the well-known Bohr model of hydrogen-like atoms. Let us consider an atom or ion with atomic number Z or nuclear charge +Ze with just one electron (charge = *-e*, mass =*m*) revolving around the nucleus in permissible circular orbits. As students calculate in higher secondary physics, in the ground state of the atom the electron moves in an orbit of radius

$$r_1 = 0.53 / Z \text{ angstrom} \quad \ldots \quad (1)$$

with a speed, $v_1 = Z\, c\, \alpha$ ... (2)

where, $\alpha = e^2 / (4\pi\, \varepsilon_0\, \hbar\, c) \approx 1/137$ ... (3)

is called the fine structure constant. In Equation (3), ε_0 is the permittivity of free space, $\hbar (= h/2\pi)$ is the reduced Planck's constant, and c is the speed of light in vacuum.

What is interesting is the dependence of orbit radius and electron speed on the atomic number *Z*. The former decreases and the latter increases with increasing *Z*. Einstein's celebrated theory of relativity stipulates that the velocity of the electron cannot be more than c. Then, taking $(v_1)_{max} = c$ in Equation (2), we get an upper limit of *Z* namely

$$Z_{max} = 1/\alpha \approx 137 \quad \ldots \quad (4)$$

Equation (4) may be interpreted by saying that one cannot have stable atomic structure with a nuclear charge in excess of 137e.

Now, from Equations (4) and (1) we get the ground state radius of the corresponding atom or ion

$$r_{1\,min} = 0.53 / Z_{max}$$

$$= 0.004 \text{ angstrom} = 400 \text{ fermi} \ldots \quad (5)$$

Note that 1 *angstrom* $=10^{-10}$ m and 1 fermi= 10^{-15}m. Atomic size is usually measured in angstrom unit whereas nuclear size is measured in fermi unit.

Both Equations (4) and (5) may justify why we cannot have arbitrarily many naturally occurring elements and there may be an upper limit to the number.

It may also be noted that the value of α as determined in Equation (3) is more or less an exact value obtained by substituting the known values of e (= 1.6×10^{-19} C), $1/4\pi\varepsilon_0$ (= 9×10^{9} N m^2 C^{-2}), $\hbar$ (= 1×10^{-34} $J\,s$), c (= 3×10^{8} $m\,s^{-1}$). A rough mental calculation may be very quickly done by taking the order of magnitude of the above quantities: e (= 10^{-19} C), $1/4\pi\varepsilon_0$ (= 10^{10} $N\,m^2\,C^{-2}$), $\hbar$ (= 10^{-34} $J\,s$), c (= 10^{8} $m\,s^{-1}$) and this yields the value of $\alpha \approx 10^{-2}$, which is often used as an order-of-magnitude estimate of the fine structure constant.

3. Size of the Universe

Apparently, based on the work of a previous astronomer called Aristarchus, Archimedes guessed a formula (Venkataraman, 2013, p. 107) according to which the ratio of the size of the universe and the size of the solar system is equal to the ratio of the size of the solar system and the size of the earth. Assuming them all to be spherical in shape we get,

$$\textit{diameter of the universe} = \frac{(\textit{diameter of the solar system})^2}{\textit{diameter of the earth}} \qquad \ldots(6)$$

Now, using the present value of the diameter of the solar system as 287.46 billion kilometres (https://www.universetoday.com/15585/diameter-of-the-solar-system/) rounded up to 3 x 10^{14} km and the average diameter of the earth as 12, 742 kilometres rounded up to 1.3 x 10^{4} km, we get from Equation (6),
diameter of the universe=7 × 10^{24} km
which appears to be just about 8 times the current value of the diameter of the observable universe, 8.8 x 10^{23} km (https://en.wikipedia.org/wiki/Observable_universe). Thus, the simple looking Archimedes formula not only gives a plausible estimate of the size of the universe thereby illustrating the utility of back-of-the-envelope estimate but also sheds light on the structure of the universe.

4. Planetary Distances from the Sun

In 1772, a German astronomer Johann Elert Bode gave a simple rule (a thumb rule, sometimes loosely called a law) (*The New Book of Popular Science*, Vol. 1, pp. 84-85; https://www.britannica.com/science/Bode's-law) to calculate planetary distances from the sun. The rule may be described in a few simple steps:

(i) Start with the numbers 0 and 3.

(ii) Continue the series by doubling the last number to get the series 0, 3, 6, 12, 24, 48, 96, 192, 384....

(iii) Add 4 to each number in the series.

(iv) Divide the resulting numbers by 10 to get the planetary distances from the sun in a.u. where a.u. or astronomical unit is the distance of the earth from the sun, nearly 1.5.

Table 1: Bode's values of planetary distances in comparison with their measured values, for the solar system

Planet/ Member of solar system	Distance from Sun in a.u. (Bode's law)	Distance from Sun in a.u. (measured)
Mercury	0.4	0.38
Venus	0.7	0.72
Earth	1.0	1.0
Mars	1.6	1.5
Ceres	2.8	2.76
Jupiter	5.2	5.2
Saturn	10.0	9.5
Uranus	19.6	19.2
Neptune	38.8	30.1

In Bode's time only the first 6 planets were known: Mercury, Venus, Earth, Mars, Jupiter, and Saturn. Though there was no theoretical justification of Bode's rule, it was taken seriously by the astronomers. Its results for the first four planets matched the measured values exceedingly well as the table above indicates. The fifth planet Jupiter's distance (5.2 a.u.) is identical to the sixth number in the Bode sequence and not the fifth number (2.8 a.u.) as it should have. But the astronomers were so convinced by the predictive power of Bode's law that they thought there must be a planet at 2.8 a.u. and started looking for it. Actually, in 1801 the first and the largest asteroid Ceres was discovered between Mars and Jupiter at 2.76 a.u. thereby strengthening the power of Bode's law. Another success story came in the form of Uranus which was discovered in 1781 (see Table 1). However, the last planet in the revised planetary list for the solar system namely Neptune, discovered in 1846, shows appreciable departure from Bode's prediction.

Bode's rule beautifully exhibits the power of back-of-the-envelope calculation even if it does not enjoy theoretical support.

5. How many Molecules are there in 1 Gram of Water?

There is a straight forward back-of-the-envelope answer to this question. Molecular weight of water (H_2O) is $(2\times1 + 16) = 18$. Then in accordance with Avogadro's law, 18 gram of water contains 6.022×10^{23} molecules (the Avogadro's number) under STP (standard temperature and pressure i.e. $0^{\circ}C$ or 273 K and 1 atmosphere). Hence, if n be the number of molecules in 1 gram of water under these conditions, then

$$n = 6.022 \times 10^{23} / 18 \approx 0.3 \times 10^{23} \cdots \quad (7)$$

We may also calculate the same number in another simple way. Let us illustrate it.

According to the law of equipartition of energy, each degree of freedom of a molecule has 0.5 kT energy at temperature T K, where k is the Boltzsmann's constant (= 1.38 x 10^{-23} J/K). A molecule of water (a tri-atomic molecule) has 6 degrees of freedom. Hence the thermal energy possessed by 1 gram of water at TK is equal to 3 nkT. Let us equate this energy with the latent heat of fusion of water to calculate n.

We know that at STP, 1 gram of water needs to give up 334 J of heat energy (the latent heat of fusion of water, L_f) to convert into 1 gram of ice. We therefore have the equation

$$3 \times n \times 1.38 \times 10^{-23}\, J/K \times 273\, K = 334\, J$$

which gives, $n \approx 0.3 \times 10^{23}$ ... (8)

almost the same value as in Eq. (7). Equality between these two calculations is expected as both correspond to STP.

6. How many Atoms are there in a Human Body?

Though a large variety of atoms occur in nature, only a few of them are found in our body. According to an estimation, as percentage of body mass, they are 65 per cent oxygen (O), 18 per cent carbon *(C)*, 10 per cent hydrogen *(H)*, 3 per cent nitrogen *(N)*, 1.4 per cent calcium *(Ca)*, 1.1 per cent phosphorus *(P)* the rest being trace elements of very low abundance. The first four elements alone add up to 96 per cent of body mass and only these may be considered for our back-of-the-envelope calculation. Let us consider a 40 kg child. Then in the child's body, masses of O, C, H, and N are m_o = 26 kg, m_c = 7.2 kg, m_H = 4 kg, and m_N = 1.2 kg respectively. Now noting that their atomic weights are A_0 = 16, A_c = 12, A_H = 1, and A_N = 14 respectively and using the Avogadro's law with the Avogadro's number as $N_A = 6.022 \times 10^{23}$ for 1 gram-atom or 6.022×10^{26} for 1 kilogram-atom, we get

n_0 = the number of O atoms = $N_A \times 26/16 \approx 1.0 \times 10^{27}$

n_C = the number of C atoms = $N_A \times 7.2/12 \approx 0.4 \times 10^{27}$

n_H = the number of H atoms = $N_A \times 4/1 \approx 2.4 \times 10^{27}$

n_N = the number of N atoms = $N_A \times 1.2/14 \approx 0.5 \times 10^{26}$

Adding up these numbers we get the total number of atoms in the body of the 40 kg child as about 4×10^{27}. Note that in the final count the contribution of N is almost negligible compared to the other three. Hence we might have omitted N from the estimation altogether.

7. How many Protons, Neutrons and Electrons are there in a Human Body?

Out of hundreds of (elementary) particle types discovered and catalogued, only three, namely, protons, neutrons, and electrons constitute atoms. Of these protons and neutrons constitute the nucleus of atom and electrons are found outside the nucleus revolving around it. It is therefore worthwhile to find how many of these particles might be present in a human body. To get an answer let us again consider the case of a 40 kilogram child with the four dominant elements O, C, H and N as in the previous example. The numbers of the respective atoms in the body of the child n_O, n_C, n_H and n_N as calculated there shall be used here. Now, in view of the facts that (i) atomic weight of an element is equal to the total number of protons and neutrons in its nucleus, (ii) number of electrons is equal to that of protons because of charge neutrality of atom, and (iii) the number of protons is equal to that of neutrons in the lighter elements considered here except H which has only one proton and no neutron, it is enough to calculate the number of protons in each case. Accordingly, the number of protons in $O = n_O \times (A_O/2) \approx 8.0 \times 10^{27}$

the number of protons in $C = n_C \times (A_C / 2) \approx 2.4 \times 10^{27}$

the number of protons in $H = n_H \times (A_H) \approx 2.4 \times 10^{27}$

the number of protons in $N = n_N \times (A_N / 2) \approx 0.3 \times 10^{27}$

Adding these numbers we get the total number of protons in the body of the 40kg child as nearly 1.3 x 10^{28}, which is also equal to the number of electrons in the body, whereas the number of neutrons is nearly 1 x 10^{28} as there are no neutrons in H.

This back-of-the-envelope calculation gives an idea about the number of elementary particles like proton, neutron, and electron in a human body.

Concluding Observations

In view of the utility of back-of-the-envelope calculations to arrive at quick answers, make plausible predictions, develop scientific thinking and broaden perspectives, we have discussed a few examples keeping them simple. More examples, at a higher level, may be found in the literature, a glimpse of which is available in the references. The basic process involves some reasonable guesses or assumptions, approximations, rounding up, etc., which are considered to be part of the skill set of a person doing science. Hence teachers may like to train the students in the process with suitably designed exercises. An interesting question may initiate the process and guidance may be provided by the teacher as and when needed. A simple example of guesstimating the perimeter and area of the school premises or even the classroom may be a good starting point at an early stage.

It may also be noted that in many of the cases discussed here, the numbers involved are usually large. It could mean that large numbers are more amenable to approximations than small numbers. This is similar to the fact that the same error is more tolerable in the measurement of a large quantity than in the measurement of a small quantity. For example, you may not notice an error of 1 mm in the length of your shirt but a similar error in the size of a ring may pose problem. Moreover, such calculations are likely to challenge the mental faculty of the children and make them wiser. Such practices are often followed by professionals in all fields and, therefore, those who are on their way to join them ought to be educated in the same.

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