# Synergistic Relationship Between Computational and Mathematical Thinking: The Role of Spreadsheets

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**Abstract**

*There is widespread acknowledgement that computational thinking needs to be integrated in the school curriculum. Mathematical thinking (MT) and computational thinking (CT) are mutually supportive and yet distinct. Mathematics as a fundamental school subject therefore becomes the likely choice for integrating CT practices. However, the nature of tasks that integrate MT and CT remains elusive. This paper presents an exemplar from a larger study, where fractal explorations and simulation of problems in probability enabled by spreadsheets led pre-service teachers to develop mathematical concepts and also engage with CT practices. The tasks enabled the participants to engage in processes of visualization, generalization, recursion, iteration, and analysing algorithms, which are important from computational and mathematical standpoints. The evidence of student learning indicates a compelling argument for incorporating such tasks in foundational mathematics courses in pre-service teacher education programmes.*

***Keywords:*** *Computational thinking, mathematical thinking, teacher education programmes, spreadsheets*

## Introduction

Computational thinking (CT) is considered to be a skill set that is applicable across contexts and domains. The need to integrate CT in K–12 curricula has been widely acknowledged. Pei et al. (2018) have noted that advancement in computing technology in the past few decades has made it imperative "to make computational thinking a part of every student's educational experience." (Pei et al., 2018, p. 75). In light of the increasing role of technology in education, the PISA 2021 Mathematics Framework recognises that mathematical literacy should leverage the "synergistic and reciprocal relationship between mathematical thinking and computational thinking" (OECD, 2021, p. 7).

CT and mathematical thinking (MT) are closely related to each other. Some of the skills common to CT and MT include the ability to engage with challenging problems, reduce problems to simpler and more tractable versions, represent concepts and solutions in computationally meaningful ways, making abstractions and generalisations and engage with multiple paths of inquiry while solving problems. Thus, mathematics as a fundamental school subject is the natural choice for integrating CT-based activities. The National Education Policy 2020 emphasises this in its policy document by stating that "mathematics and computational thinking be given increased emphasis throughout the school years." It recommends that CT be integrated into the mathematics curriculum through innovative methods such as using puzzles and games "that make mathematical thinking more enjoyable and engaging." (p. 15)

However, developing tasks that integrate both CT and MT is not straightforward and remains a major pedagogical challenge. Teacher preparation programs (TEPs), both in-service and pre-service, need to address this challenge. In general, foundational mathematics courses in pre-service TEPs emphasise school mathematical content and pedagogy but often fail to offer adequate opportunities for engaging with CT. In this article, we shall make a case for incorporating CT-based activities in the mathematics content courses in TEPs. We argue that pre-service teachers can develop CT skills if they engage in appropriately designed tasks that illustrate the practical significance of mathematics as a discipline and also enable them to apply mathematical concepts to real-world phenomena via computer-based explorations. We propose that spreadsheets can provide the appropriate learning environments for fostering MT and developing CT practices. Spreadsheets are easily accessible, do not require extensive knowledge of coding, and have low technical overheads as compared to programming languages. This article describes a study where CT- MT tasks enabled by spreadsheets, designed to elicit MT and computational skills, were integrated into a foundational mathematics course taught in the first year of an undergraduate four-year pre-service TEP. The aim of the course is to revisit school mathematics from a higher standpoint through in-depth exploration of the concepts and content. The author, who taught the course, selected 34 pre-service teachers from among the 67 who were enrolled in the TEP. These pre-service teachers had volunteered to participate in the study and had studied mathematics up to grade 12. Their prior knowledge included secondary-level mathematical topics, such as algebra, probability, trigonometry, and coordinate geometry. They were also familiar with the basic features of Excel, such as inserting formulas in cells and using simple commands. The CT- MT tasks were designed by the author and covered a wide variety of topics, from fractal explorations to simulation of problems based on probability.

## Theoretical Framework

Researchers have interpreted CT and its connection with mathematics in different ways. The pioneering work by Papert (1980) and his colleagues led to concretizing the term CT. Papert believed that the primary goal of CT was to build new ideas. His theory of constructionism, considered to have arisen out of Piaget's theory of constructivism, posits that learning can be enhanced if the learner engages in "creating a meaningful product." In his book Mindstorms, he envisioned the computer as a "mathematics-speaking being" and delved into how constructionist math environments could integrate MT and CT.

While the term CT was concretised by Papert, the credit for popularising it may be alluded to Jeanette Wing. She described CT as a "fundamental skill for everyone" and advocated that CT be given as much importance as reading, writing, and arithmetic in a child's education. (Wing, 2006, p. 33). According to her, CT comprises four primary components: decomposition, pattern recognition, abstraction, and algorithmic design. Decomposition is the process of reducing a mathematical problem into more tractable sub-problems. Pattern recognition involves identifying commonalities, structures, and regularities within mathematical data. Abstraction deals with generalizing patterns into mathematical results or formulae. Algorithm design refers to creating a sequence of steps that, when exercised by a machine (computer) or a human, can lead to the solution of the problem. While these fundamental components provide a clear understanding of CT with mathematics learning, incorporating these in mathematics lessons is not straightforward. To address the challenge of meaningfully integrating CT in mathematics, it would be helpful to have a framework for guiding teachers and practitioners to enable their students to engage in CT in the mathematics classroom. Ho et al. (2019) propose a framework comprising four design principles, based on the fundamental components of CT, to enable teachers to interpret them for the instructional design of CT-MT tasks. These are framed in the form of questions.

Complexity Principle: Does the mathematical concept on which the task is based lead to an adequately complex problem? Is the problem worth solving, and can it be decomposed into simpler sub-problems?

This refers to the idea that the mathematical task should be challenging enough so that finding its solution is a worthwhile pursuit. In other words, the task should be sufficiently complex so that it necessitates decomposition. If it is too simple and has an easy solution, then it would not need to be decomposed or reduced to a simpler form.

Data Principle: Can the concepts be observable through data? Can the data be collected or created and then analysed?

This entails collecting or generating data from which patterns or structures may be identified.

Mathematics Principle: Can the problem be expressed in mathematical terms, and is a mathematical solution possible?

This refers to the mathematical formulation of the problem. A real-life problem or phenomenon needs to be represented mathematically so that it can be solved by mathematical means.

Computability Principle: Does the problem lend itself to exploration? Is it programmable? Is it amenable to solutions via algorithms and computer programming?

This principle focuses on arriving at a solution either by a machine (computer) or a human being.

Other researchers, such as Weintrop et al. (2016), have also attempted to explicate computational thinking, especially in the context of STEM disciplines. They advocate a classification that includes four kinds of practices: those associated with handling data, modelling and simulation, computational problem solving, and systems thinking. Their article presents a sequence of lesson plans that integrate CT in high school science classrooms. Despite attempts by researchers, a lack of agreement concerning how CT can be integrated into the mathematics classroom remains a challenge. However, it is imperative to orient future mathematics teachers to the various aspects of CT in their mathematics content courses so that they are equipped to meaningfully incorporate CT in their teaching. This article is therefore a beginning step in exploring the integration of CT-MT tasks in foundational mathematics courses in TEPs and to enable pre-service teachers to accrue the benefits of engaging with such tasks.

## CT – MT Tasks

In this section, we will briefly describe two CT- MT tasks, one based on fractal explorations and the other on simulating the Birthday Paradox, which was assigned to the 34 participants in the study. These tasks were administered as a part of a larger module comprising five CT-MT tasks whose design was informed by the framework proposed by Ho et al. (2019). The tasks were implemented by the author in the classroom through reading materials, worksheets, and whole-class discussions. The worksheets comprised step-by-step investigations, which enabled the participants to explore the concepts. For computational work, most of the participants used MS Excel on their mobile phones, while a few brought their laptops to class. The fractal explorations took ten hours of classroom time, while the Birthday Paradox exploration took three hours. Post the completion of the tasks, the participants submitted their worksheets for evaluation and also documented their explorations in a file for their own records.

### Fractal explorations

The topic of fractals can be an excellent resource for engaging with ideas of recursion, iteration, self-similarity, and fractal dimension, all of which are important for developing MT and CT. The investigatory tasks in the worksheet required the pre-service teachers to explore various attributes, such as length, perimeter, and area of the Sierpinski Triangle, Koch Curve, and Koch Snowflake and also create their own fractal patterns. The spreadsheet MS Excel was used as a vehicle for exploration. While investigating the Sierpinski Triangle (see Figure 1), participants predicted the geometric sequences, which emerged from computing the number of black triangles and the combined black area at each stage. Across stages, the number of black triangles led to the sequence $1,3,3^2,3^3,\ldots$ whereas the shaded area followed the sequence 1, 3/4, $(3/4)^2\ldots$ Recursive and explicit formulae for these attributes were derived for the *n*th stage as $Sn = 3Sn{-}1$, $Sn= 3n$ and $An = (3/4)$ $An{-}1$, $An = (3/4)^n$ respectively. At this point, the teacher asked the participants to reflect and comment on the two types of formulae. One participant responded, "The explicit formula is more useful as we can get the exact number of black triangles at any stage by inserting the value of *n*. This does not happen with the recursive formula. So, $S_5 = 3S_4$ but I can't calculate $S_5$ unless I know $S_4$. But if I use the explicit formula, $S_5 = 3^5$, I get the exact number."

Another student felt that the recursive formulae were also useful as they "highlighted an important aspect of the structure of the Sierpinski triangle." Post this discussion, the participants generated numerical and graphical representations of the two sequences on a spreadsheet. This led to an interesting classroom discussion as one participant explained, "At least numerically and graphically, the spreadsheet helps us to visualize the Sierpinski triangle in higher stages. It is impossible to understand the growth of the fractal using only pictorial representations." Another participant concurred, "As the number of stages increases beyond 3, the Sierpinski triangle becomes too complicated as many small equilateral triangles get added at every stage." When asked by the teacher to comment on the graphical representations of the number of shaded triangles and the shaded area (Figure 1), one participant observed "the graphs show opposite processes—growth as well as decline." The teacher used this opportunity to enable participants to conclude that with an increase in the number of stages, the number of black triangles grows exponentially, whereas their combined area approaches 0. It was heartening to see the participants engage with different modes of representation of the Sierpinski triangle while at the same time acknowledging the benefits and disadvantages of each mode. Figure 1 illustrates how the fractal was explored using multiple representations—numerically, graphically, and pictorially.

The notion of self-similarity was also explored within the different stages of Sierpinski triangle construction. This entailed identifying copies of earlier stages in later stages of the fractal. For example, in stage 1 of the Sierpinski triangle, three copies of stage 0 can be seen, in stage 2, three copies of stage 1, and so on. One participant commented, "The idea of self-similarity can be used to build higher stages from lower ones. So I can use three copies of stage 3 to build stage 4 and so on." This observation was an important milestone in the Sierpinski triangle exploration.

| Stage No. | No of shaded Triangles | Shaded area |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 3 | 0.75 |
| 2 | 9 | 0.5625 |
| 3 | 27 | 0.421875 |
| 4 | 81 | 0.31640625 |
| 5 | 243 | 0.23730469 |
| 6 | 729 | 0.17797852 |
| 7 | 2187 | 0.13348389 |
| 8 | 6561 | 0.10011292 |
| 9 | 19683 | 0.07508469 |
| 10 | 59049 | 0.05631351 |
| 11 | 177147 | 0.04223514 |
| 12 | 531441 | 0.03167635 |
| 13 | 1594323 | 0.02375726 |
| 14 | 4782969 | 0.01781795 |
| 15 | 14348907 | 0.01336346 |

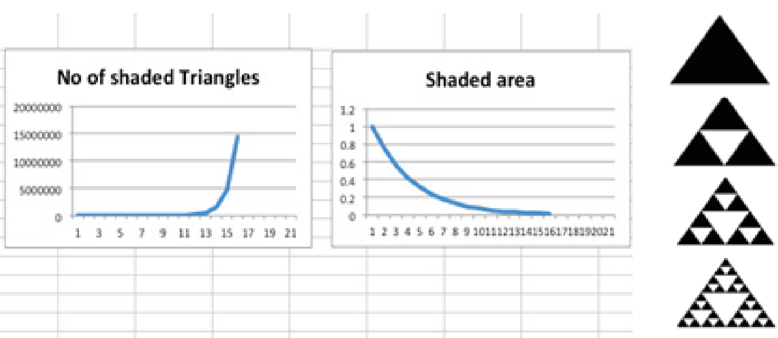


*Figure 1: Exploring the Sierpinski triangle fractal using multiple representations*

While exploring the Koch snowflake in Excel, the participants were required to find the total area enclosed by the snowflake. It came as a revelation that as n increases, the number of stages increases, the additional area at each stage keeps decreasing, and the total area approaches a fixed value of 2√35 ≈0.69 (as shown in figures 2 and 3). The area inside the snowflake was also computed using the theory of geometric sequences as follows:

$$\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\times\frac{1}{3}\left\{1+\frac{4}{3^2}+\frac{4^2}{3^4}+......\right\}=\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\times\frac{1}{3}\times\frac{9}{5}=\frac{\sqrt{3}}{4}\left(1+\frac{3}{5}\right)=\frac{\sqrt{3}}{4}\times\frac{8}{5}=\frac{2\sqrt{3}}{5}$$

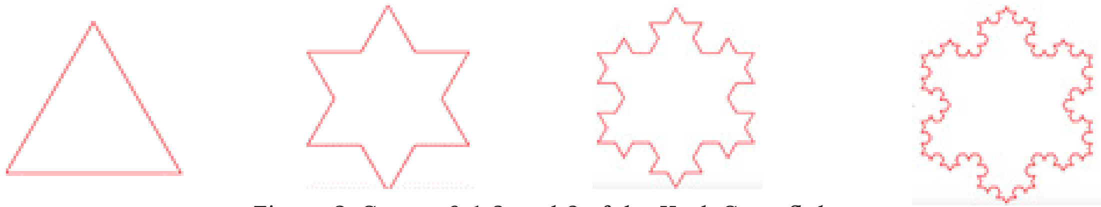

*Figure 2: Stages 0,1,2 and 3 of the Koch Snowflake*

Some participants expressed that although they had studied the topic of geometric sequences in school, it seemed "very abstract." However, the fractal explorations helped them to see the relevance of the topic. One commented, "We had learned so many formulae on the topic of geometric sequence. Now I know why they can be used.

Figure 3 illustrates the numerical and graphical exploration of the Kock snowflake in Excel"

| stage | line segments | length of line segment | perimeter | Additional area | Total area |
|---|---|---|---|---|---|
| 0 | 3 | 1 | 3 | 0 | 0.433012702 |
| 1 | 12 | 0.333333333 | 4 | 0.144337567 | 0.577350269 |
| 2 | 48 | 0.111111111 | 5.333333333 | 0.06415003 | 0.641500299 |
| 3 | 192 | 0.037037037 | 7.111111111 | 0.028511124 | 0.670011424 |
| 4 | 768 | 0.012345679 | 9.481481481 | 0.012671611 | 0.682683034 |
| 5 | 3072 | 0.004115226 | 12.64197531 | 0.005631827 | 0.688314861 |
| 6 | 12288 | 0.001371742 | 16.85596708 | 0.002503034 | 0.690817896 |
| 7 | 49152 | 0.000457247 | 22.47462277 | 0.00111246 | 0.691930355 |
| 8 | 196608 | 0.000152416 | 29.96616369 | 0.000494427 | 0.692424782 |
| 9 | 786432 | 5.08053E-05 | 39.95488493 | 0.000219745 | 0.692644527 |
| 10 | 3145728 | 1.69351E-05 | 53.2731799 | 9.76645E-05 | 0.692742191 |
| 11 | 12582912 | 5.64503E-06 | 71.03090654 | 4.34064E-05 | 0.692785598 |
| 12 | 50331648 | 1.88168E-06 | 94.70787538 | 1.92918E-05 | 0.69280489 |
| 13 | 201326592 | 6.27225E-07 | 126.2771672 | 8.57411E-06 | 0.692813464 |
| 14 | 805306368 | 2.09075E-07 | 168.3695562 | 3.81072E-06 | 0.692817274 |
| 15 | 3221225472 | 6.96917E-08 | 224.4927416 | 1.69365E-06 | 0.692818968 |
| 16 | 12884901888 | 2.32306E-08 | 299.3236555 | 7.52734E-07 | 0.692819721 |
| 17 | 51539607552 | 7.74352E-09 | 399.0982074 | 3.34549E-07 | 0.692820055 |
| 18 | 2.06158E+11 | 2.58117E-09 | 532.1309432 | 1.48688E-07 | 0.692820204 |
| 19 | 8.24634E+11 | 8.60392E-10 | 709.5079242 | 6.60837E-08 | 0.69282027 |
| 20 | 3.29853E+12 | 2.86797E-10 | 946.0105656 | 2.93705E-08 | 0.6928203 |

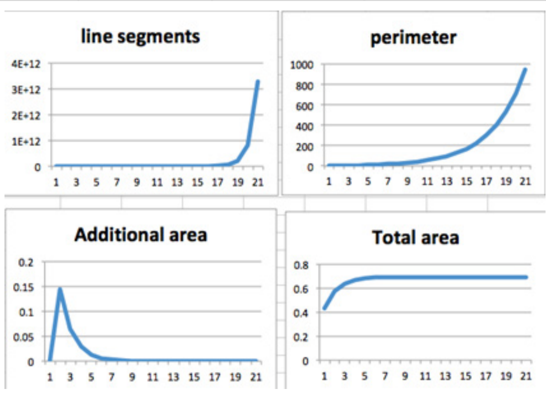


*Figure 3: Numerical and graphical exploration of the Koch Snowflake construction in Excel*

Exploration of the Koch curve (Figure 4) led to a discussion on matrix transformations. Students were scaffolded to obtain a sequence of four transformations, $f1$, $f2$, $f3$, and $f4$, which transition the Koch curve from stage n to stage n+1. The first step was to visualize the transition from stage 0, comprising a line segment of unit length, to stage 1, where the segment is trisected, the middle third removed, and an equilateral triangle is raised on the middle third. To achieve this, students used matrices to perform the transformations of shrinking, shifting, and rotating the line segment.

$$f_1 X = \left[\frac{1}{3}\,0\;\;0\,\frac{1}{3}\right] X,$$

$$f_2 X = \left[\frac{1}{6} - \frac{\sqrt{3}}{6}\,\frac{\sqrt{3}}{6}\,\frac{1}{6}\right] X + \left[\frac{1}{3}\;\;0\right],\; f_3 X = \left[\frac{1}{6}\,\frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{6}\,\frac{1}{6}\right] X + \left[\frac{1}{2}\,\frac{0\sqrt{3}}{6}\right],\; f_4 X$$

$$= [1/3\;0\,0\;1/3]\,X + [2/3\;\;0]$$

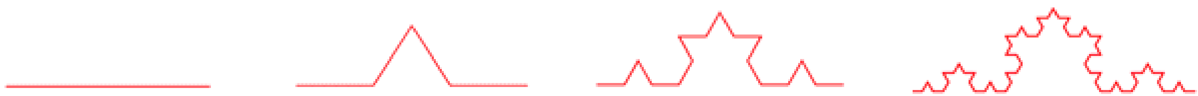

*Figure 4: Stages 0,1,2 and 3 of the Koch Curve*

## Simulating the Birthday Paradox

The concept of probability is introduced and developed at the secondary school stage and forms an integral part of the curriculum. The experiments used for illustrating the fundamental concepts in textbooks tend to be restricted to coin tossing, die rolling, or selecting cards from a deck of cards. While these may be useful as examples, they do not relate the concept of probability to real-life phenomena, thus rendering the topic dull. Interesting problems, such as the Birthday Paradox, leading to meaningful explorations can truly enliven the teaching of probability. Such problems provide a context to introduce the concept of simulation, a method for mimicking random behaviour or phenomena through the use of computers. In this section, we shall illustrate how the pre-service teachers explored the birthday paradox by first mathematizing the problem and then simulating it on MS Excel.

They were presented with the following problem in the worksheet:

How many people do you need to bring together to ensure that there are at least two people with the same birthday? Here birthday refers to birth date and month.

Most students were unsure of the answer. However, the immediate response from a few participants was 367. One explained, “There are 366 possible distinct birthdays, including 29th February (in a leap year). If one more date is added to this list, it would coincide with one of the 366 dates.” However, when they were told that the chance of a birthday match in a group of 60 people was almost certain, their reaction was one of disbelief. This claim, popularly known as the birthday paradox, takes everyone by surprise. To verify this, an experiment was conducted, wherein 60 birthdays (of friends and relatives) were noted by the participants on slips of paper. These were folded and placed in a box. The box was circulated among the participants, wherein each participant selected a slip and recorded the date on it. The box was circulated till a match was found. The same experiment was repeated with another set of 60 birthdays, which again led to a match. Thinking this to be a coincidence, many students remained sceptical. However, everyone agreed that it was impractical to keep listing birthdays to check for a match. For the teacher, this was an opportune moment to introduce the concept of simulation on a spreadsheet. The idea was to randomly generate 60 birthdays on Excel and check if the list had a repeated date. Several simulations could be generated very quickly in a matter of seconds. In each simulation, the first step was to generate a column of 60 integers between 1 to 12 (both inclusive, as shown in column B of Figure 5) representing the months (1, 2, 3, etc. representing January, February, March, etc.). To achieve this, students were familiarized with the command RANDBETWEEN(1, *n*), which randomly generates an integer between 1 and *n*. Hence RANDBETWEEN(1,12) was used to create a column of months. The next step was to use RANDBETWEEN(1,31) to generate a column of 60 integers between 1 and 31 (both inclusive) to indicate the day of the month (column C of Figure 5). The data in columns B and C represent a set of 60 (randomly generated) birthdays. For example, a 10 in column B and 31 in column C represent the date 31st October. Scrolling through this list of dates to search for a match was time-consuming. To simplify the process, the operation =100*B1+C1 was used to convert the dates to three- or four-digit numbers and store them in column D. Once this was done, the first one or two digits of the numbers (in the column called Bday) represented the month, and the last two digits represented the day of the month. Thus, 806 in the list represents 6th of August, while 1113 represents the 13th of November. This list of dates could now be copied to a different column and sorted so that the repeated date (birthday match) appeared in two consecutive rows, making it easily identifiable. In Figure 5, after sorting, the repeated birthday is 428, that is, 28th April (column F), and 318, that is, 18th March (column H). The simulation was repeated several times to confirm that there is indeed a match every time a set of 60 birthdays is randomly generated. This aroused the curiosity of the participants of the study.

| A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| S. No | Month | Day | Bday | | Bday list1 | | Bday list2 |
| 1 | 6 | 13 | 613 | | 115 | | 103 |
| 2 | 3 | 6 | 306 | | 119 | | 114 |
| 3 | 1 | 12 | 112 | | 204 | | 120 |
| 4 | 3 | 6 | 306 | | 208 | | 124 |
| 5 | 10 | 31 | 1031 | | 215 | | 202 |
| 6 | 4 | 13 | 413 | | 215 | | 213 |
| 7 | 5 | 15 | 515 | | 221 | | 215 |
| 8 | 3 | 23 | 323 | | 228 | | 219 |
| 9 | 8 | 6 | 806 | | 229 | | 221 |
| 10 | 1 | 18 | 118 | | 302 | | 222 |
| 11 | 12 | 9 | 1209 | | 311 | | 228 |
| 12 | 8 | 24 | 824 | | 314 | | 315 |
| 13 | 7 | 5 | 705 | | 318 | | 318 |
| 14 | 3 | 26 | 326 | | 320 | | 318 |
| 15 | 11 | 13 | 1113 | | 321 | | 320 |
| 16 | 3 | 8 | 308 | | 324 | | 329 |
| 17 | 7 | 16 | 716 | | 410 | | 329 |
| 18 | 5 | 24 | 524 | | 417 | | 404 |
| 19 | 4 | 15 | 415 | | 424 | | 405 |
| 20 | 3 | 16 | 316 | | 428 | | 502 |
| 21 | 5 | 18 | 518 | | 428 | | 510 |
| 22 | 6 | 15 | 615 | | 501 | | 511 |

*Figure 5: Birthday Problem simulation in MS Excel*

The simulation was used to verify the claim that if we randomly gather 60 people, the probability of a birthday match is a certain event. One participant observed, "Each time we simulate a list of 60 birthdays, we see a match... Sometimes there is more than one match." But another commented, "Simulation does not help prove the claim. It needs to be justified mathematically." This was indeed a high point for the teacher, who realized that the pre-service teachers were not satisfied merely with a numerical verification but were looking for proof. The latter requires the problem to be analyzed using probability theory. This led to another interesting class discussion where students first computed the probability of a birthday match among three, four, and five people, respectively. They observed the patterns within these expressions and tried to arrive at a generalization for n people. The probability that all three people will have distinct birthdays in a group of three people is given by the expression

$$\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}.$$

Hence, the probability, that in a group of three people, *at least* two have the same birthday is given by

$$1-\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}=\mathbf{1}-\frac{365\times 364\times 363}{365^3}.$$

Extending this argument to four people, the probability of finding a birthday match is

$$1-\frac{365\times 364\times 363\times 362}{365^4}$$

and similarly for five people, it is.

$$1-\frac{365\times 364\times 363\times 362\times 361}{365^5}.$$

Students observed the pattern within these expressions and generalized the result to obtain the expression

$$1-\frac{365\times 364\times 363\times ......\times(365-(n-1))}{365^n}$$

which represents the probability of a birthday match among *n* people. Using factorial notation this was further simplified to: As *n* (the group size)

$$1-\frac{365!}{(365-n)!\times 365^n}.$$

approaches 60, the value of this expression approaches 1. The high point of this exploration was the fact that students were now convinced that the paradox was indeed true.

## Discussion And Conclusion

Based on the "synergistic relationship between CT and MT" as espoused by the PISA mathematics framework, the study described in this article attempted to explore the potential of incorporating spreadsheet-enabled CT-MT tasks in a foundational mathematics course in a pre-service TEP. In the study, 34 pre-service teachers engaged in modules comprising CT-MT tasks. The tasks were facilitated through reading materials, worksheets, and classroom discussions. The choice of the tasks was based on students' knowledge of mathematical content and the four design principles proposed by Ho et al. (2019). Here we shall explain how these design principles informed the CT-MT tasks on fractal explorations and the Birthday Paradox, which were open-ended and investigatory.

**Complexity Principle**: Fractals are complex objects that represent infinite recursive processes. However, observing their constructions in the initial stages can lead to an understanding of the nature of their growth. In the study, investigating attributes such as the length, perimeter, and area of the fractals in the initial stages led to geometric sequences, identifiable patterns, and their generalization. The Birthday Paradox, a real phenomenon, makes a claim that defies intuition. Yet exploring the probability of a birthday match in small group sizes (of three, four, and five people) helped us understand the problem, making it more tractable.

**Data Principle**: The data involved in the fractal explorations were numerical values of the attributes in the different stages of the fractals. Similarly, the data required in the Birthday paradox was the random generation of birthdays, which could be obtained by collecting the birthdays of actual people or by simulation on a spreadsheet. In both explorations, the spreadsheet played a crucial role by quickly generating the data, thus enabling the participants to focus on observing patterns.

**The mathematical principle**: The mathematical principle allows for the generalization of the patterns and regularities found in the generated data. This led to representing the attributes of the fractals mathematically through recursive and explicit formulae. In the Birthday paradox exploration, the patterns within the expressions of the probabilities of a birthday match for smaller group sizes led to a more general formula for a group of any size. The general formula then led to proof of the paradox.

**Computability Principle**: The recursive and explicit formulae of the fractal attributes could be easily incorporated into a spreadsheet to generate numerical values of the attributes at higher stages. Further graphical representations led to a deeper insight with regard to the nature of fractal growth. In the Birthday Paradox exploration, the simulation generated birthdays randomly to check for a match. Thus, the four design principles helped determine the suitability of these problems as CT-MT tasks.

An in-depth analysis of the participants' responses revealed that the described tasks enabled them to engage in MT and CT practices. We summarize them here briefly.

**Problem decomposition**: To begin with, the participants explored the attributes of the fractals in the initial stages (0 to 4) through pictorial representations. This led them to observe important patterns in fractal growth. For example, in the Sierpinski triangle, a geometric sequence comprising powers of 3 emerged while counting the number of black triangles across stages, whereas in the Koch curve, the number of line segments led to powers of 4. Similarly, while exploring the birthday paradox, the pre-service teachers began by collecting sets of 60 birthdays to search for a match. However, to analyze the problem, they used the definition of probability to work out the probabilities of a birthday match in groups of 3 and 4 people. In both tasks, simplifying the problem led to a starting point for exploration.

Identifying geometric sequences within fractal attributes entails MT, whereas observing their recursive self-similar structures requires CT. In the Birthday Paradox task, visualizing the simulation steps on the spreadsheet requires CT, while deriving the expressions of the probability of a birthday match requires MT.

**Pattern recognition and generalization**: The participants worked with the geometric sequences arising out of the attributes of the fractals and generalized these patterns symbolically by deriving mathematical formulae, both recursive and explicit. Also, identifying attributes of the nth stage in terms of the (n-1)th stage or using copies of the (n-1)th stage to build the nth stage led to understanding self-similarity and fractal dimension. This enabled them to visualize recursion while also expressing it symbolically. In the birthday problem, observing patterns in the expressions of probabilities of birthday matches in small group sizes provided insight into the general formula. Therefore, we generalized the identified patterns in both tasks.

**Data generation**: The pre-service teachers incorporated the recursive and explicit formulae (for attributes such as area and perimeter of the fractals) into the spreadsheet. Using this approach, they created meaningful numerical data, which further helped them to observe the properties of the fractals at the higher stages even without pictorial representations. In the case of the Birthday Paradox, simulation helped to quickly generate sets of 60 randomly generated birthdays. They were able to verify the paradox by scrolling through these data sets.

**Abstraction**: As participants worked with pictorial and symbolic representations, they made connections between geometric representations of fractals and algebraic expressions, while the spreadsheet enabled them to visualize the fractals numerically and graphically at higher stages. Here participants could see the benefits of multiple representations in action. Feedback taken from participants at the end of the fractal tasks revealed that their understanding of geometric sequences was strengthened and that they found the topic more relevant. Arriving at the formula for the probability of a birthday match for any number of people led to a symbolic representation of the birthday paradox. Further, participants were able to mathematically justify the reason for finding a birthday match among 60 people, while logical reasoning tells us that the number should be 367. Working through this task strengthened my understanding of several foundational concepts in probability theory.

Hence, these open-ended tasks enabled by spreadsheets highlight the potential for integrating CT practices in the mathematics classroom. They provided a natural context for engaging in the processes of visualization, recursion, iteration, generalization, and analyzing algorithms, which are essential from computational as well as mathematical standpoints. Further, as argued by Pea (1987), technology in the form of spreadsheets acted as an amplifier as it increased the scope of the tasks by providing access to numerical data that could not have been obtained manually. This was evident from the participants' responses during classroom discussions and feedback taken at the end of the tasks. With regard to the fractal explorations, many participants expressed that "the numerical and graphical outputs provided by the spreadsheet led to a deeper understanding of the fractal, which could not have been obtained manually or through pictorial representations." In the birthday paradox investigation, they shared, "We can't keep gathering groups of 60 people to check for a repeated date. That would be impossible. However, the spreadsheet instantly generates the dates, which aids in verifying the paradox. Technology also played the role of a reorganizer, in the sense that each task had a different trajectory of exploration. It also gave access to higher-level concepts by affording the possibility to explore the problem through multiple representations—numerical, symbolic, and graphic. In general, by working through the tasks, the pre-service teachers developed the capability of using the spreadsheet meaningfully for exploration. They also appreciated the power afforded by the spreadsheet in the investigations. While attempting the tasks, they engaged in processes such as looking for patterns and invariances within data, making conjectures, selecting between multiple representations, simplifying or generalizing aspects of the problem being explored, and identifying new opportunities for exploration. All these are important skills for developing CT and MT. About 28 participants rated the CT-MT tasks as meaningful and relevant. One participant mentioned, "These tasks are fascinating, and they make the topics so interesting. This is different from the math I learned at school." This study therefore emphasizes that appropriately designed CT-MT tasks, founded on the symbiotic relationship between MT and CT, can lead to deeper mathematical learning while fostering meaningful use of technology at the same time. Further, the positive feedback from the pre-service teachers who participated in the study and the evidence of their engagement in the tasks provide a compelling argument for incorporating such CT-MT tasks in foundational mathematics courses in TEPs.

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