THE FORGOTTEN MATHEMATICAL LEGACY OF ARYABHATA

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Aryabhata made pioneering contributions to trigonometry. The sine function finds its first mention in the Aryabhatiya (499 CE), a cryptic work written by Aryabhata when he was 23 years old. The work also indicates how to obtain the difference of sines and cosines by ingenuously using the properties of similar triangles. It then exploits these formulae to obtain the values of the sine function for angles between 0 to 90 degrees. We describe how this was done. In order to make our presentation pedagogical, we take a unit circle and radians instead of the (now) archaic notation in the Aryabhatiya and its commentaries. These seminal contributions of Aryabhata seem forgotten. This is tragic since these methods can be gainfully taught to high and higher secondary school students. We also impress upon the reader the fact that there has been a continuity in the Indian mathematical tradition, starting from the intricate geometric thinking of the Sulbastura in the Vedic era (circa 1000 BCE) to the Calculus of Madhava and his disciples (1350–1600 CE).

Keywords: Aryabhatiya, Aryabhata, ardha-jya (sine), pi, trigonometry, finite difference calculus.

Introduction

As a student of science or engineering, we have sought out the value of a trigonometric function innumerable times. Often and unthinkingly, we press down on the mouse attached to our laptop or click on our mobile calculator. The old fashioned amongst us, reach out for the bookcase which houses our table of functions. Seldom do we stop to think how this value was first generated or why were the sine and cosine functions defined. And who defined them? The credit goes to the fifth century mathematician and astronomer Aryabhata. The great master enumerated the table of sines for closely spaced angles. His methods were based on general trigonometric identities and lend themselves to extensions. The first mention of the sine function is to be found in his (one and only) seminal work the Aryabhatiya (499 CE). Aryabhata describes it in picturesque terms as the half bow-string or Ardha-Jya. This is illustrated in Fig. 1. The arrow or saar is related to the cosine function. Aryabhata was only 23 years old when he wrote his masterpiece.

The *Aryabhatiya* consists of 121 cryptic verses, dense and laden with meaning [1, 2].

The work is divided into 4 parts or *padas*: the *Gitikapada* (13 verses), the mathematics or *Ganitapada* (33 verses), the *Kalkriyapada* (25 verses) and the astronomy or *Golapada* (50 verses). The astronomy is better known. There are two verses in the mathematics

Ganitapada describing the solution of the Linear Diophantine equation. This has received due recognition. Our focus here is on the trigonometry part in the *Ganitapada* which in our opinion has suffered neglect and is a pioneering achievement of this savant.

In this article, we describe the trigonometric identities used by Aryabhata to obtain the table of sines. This entails taking the difference of the sine of two closely spaced angles and then taking the second sine difference. We follow this up with a discussion.

The Indian mathematical tradition is largely word based and in Sanskrit verses. Results are mentioned and derivations are omitted. There are no figures. The *Aryabhatiya* (499 CE) with a little over 100 cryptic, super-compressed verses of dense mathematics is a prime example. Our presentation relies on commentary of Somaiyaji Nilakantha (1444 CE – 1544 CE) [3] and the works of Shukla Kripa Shankar and K. V. Sarma [2], as well as of P. P. Divakaran [4]. But our approach is pedagogical and one which will help the student and teacher to appreciate this pioneering work. Hence, we shall take some liberties and describe the great master's work in terms of unit circle and radians instead of degrees and minutes.

The Ardha-Jya or Sine Function

As mentioned earlier the *Aryabhatiya* has some 121 verses out of which 33 verses belong to the mathematics section (*Ganitapada*). Aryabhata works with, for the first time in the history of mathematics, the sine function. It is the half-chord AP of the unit circle in Figure. 1.

$$sin(\theta) = \frac{AP}{OA} = AP \ (OA = 1)$$

The circle may be large or small; correspondingly AP and OA maybe large or small, but the L.H.S. is a function of θ and is invariant. Further, all metrical properties related to the circle can be derived using trigonometric functions and the Pythagorean theorem (also described as the Baudhayana or Diagonal theorem [4]). For example, the geometric property of a triangle can be related to the arcs of the circumscribing circle using the sine and cosine functions or the diagonals of the inscribed quadrilateral can be related to its sides. (If a recent proof of the Pythagorean theorem using the law of sines holds up to scrutiny, then all metrical properties of a circle can be obtained by trigonometry alone [5].) By emphasising the role of this half-chord Aryabhata endowed circle geometry with metrical properties. This alone may qualify him as the founder of trigonometry. But he did more.

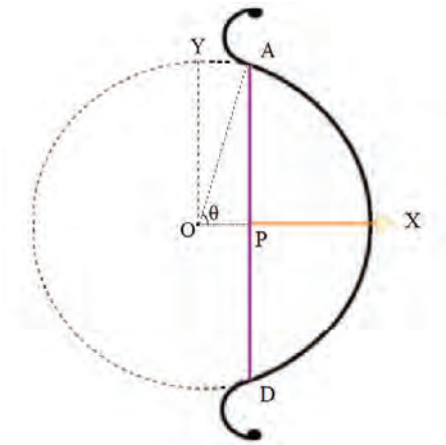

Fig. 1: The bow superposed on the unit circle. Half the bow string or half chord AP is sin(θ) as defined by Aryabhata. OP is cos(θ) while PX = 1-cos(θ) is called the saar.

It may help to note that the length of the half-chord AP is very close to the length of the arc AX only when the angle is small for example ($\sin(\theta) \approx \theta$) if theta is small and in radians). This was known to Aryabhata in all likelihood by inspection. Similarly, the sine of 90 is 1, since then the half-chord is the same as the radius. In the ninth verse of the *Ganitapada*, he uses the property of an equilateral triangle and obtains the sine of 30 as 1/2.

We paused to note that Aryabhata also states the value of π as 62832/20000 in the tenth verse. This value is 3.1416 and he is careful enough to state that this is proximate (*Asanna*) which means we can obtain better and truer values for π presumably with more effort[1].

The Difference Formula for Sine and Cosine

The verse 12 of the *Ganitapada* plays a central role in the tabulation of the sine function. It is cryptic and to unravel its meaning, we first need to obtain the difference formula for the sine. The presentation below relies on a number of sources:

(i) The commentary of Somaiyaji Nilakantha [3]

(ii) the treatment of Shukla and Sarma and

(iii) for the sake of ease of understanding we follow Divarkaran [4] and take a unit circle as opposed to a radius of 3438 by earlier workers[2].

Figure 2: depicts a quadrant of the unit circle where OX = OY = 1.
The arcs XA, XB and XC trace angles θ, $\theta + \varphi$, and $\theta - \varphi$ respectively. The half-chords AP, BQ and CR are the corresponding sine functions. We drop a perpendicular CS from the circumference onto the half-chord BQ as shown. According to his commentator Nilakantha Somaiyaji [3], Aryabhata obtained the relationship between the difference in the trigonometric functions by demonstrating that the two triangles BSC and OPA are similar. We urge the reader to try and prove this. One then has:

$$\frac{BS}{OP} = \frac{BC}{OA}$$

Now OA = 1 (unit circle), OP = $\cos(\theta)$ and BC = 2 $\sin(\varphi)$. By inspection BS = BQ – CR = $\sin(\theta + \varphi) - \sin(\theta - \varphi)$. This yields the sine difference formula (Eq. (1)

$$\sin(\theta + \varphi) - \sin(\theta - \varphi) = 2\sin(\varphi)\cos(\theta) \qquad (1)$$

The difference in the sines is proportional to the cosine of the mean angle. Similarly, one has:

$$\cos(\theta + \varphi) - \cos(\theta - \varphi) = -2\sin(\varphi)\sin(\theta) \qquad (2)$$

The difference in the cosines is proportional to the (negative) of the sine of the mean angle.

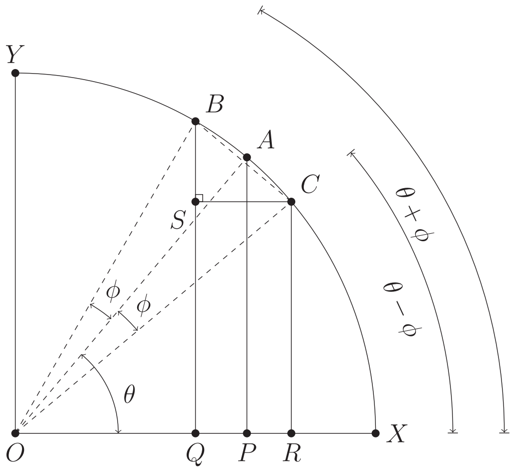

Fig. 2: Derivation of the sine difference relation. The figure depicts the quadrant of a unit circle of radii OX = OY = 1. The half-chords AP, BQ and CR are $\sin(\theta)$, $\sin(\theta + \varphi)$ and $\sin(\theta - \varphi)$ respectively. It is worth noting that (later) we shall take ϕ to be a small angle.

[1] The word Asana or proximate is to be distinguished from sthula which is approximate or roughly equal.
[2] One radian is 3438 minute

The Sine Table

Aryabhata obtained the values of the sines at fixed angles between 0 and $\pi/2$ thus, generating the sine table for $\pi/48 = 3.75$ degrees, 7.5 degrees up to 90 degrees. This table is stated in verse 10 of the first chapter, the *Gitikapada*. The table has been used by Indian astronomers (and astrologers) in some form or another since 499 CE up to the present. We shall see how the table was generated.

Let us take $\varphi = \epsilon/2$ where ϵ is small. We take $\theta = (n - 1/2)\epsilon$ where n is a positive integer from 1 to N. To fix our ideas $\epsilon = \pi/48 = 3.75° = 225'$ and N = 24. We re-state the sine and cosine difference formulae (Eqs. (1) and (2)) from the previous section:

$$\delta s_n = s_n - s_n-1 = 2s_{1/2}c(n-1/2) \qquad (3)$$

$$\delta c_n = c_n - c_n-1 = -2s_{1/2}s(n-1/2) \qquad (4)$$

where the symbol sn stands for $\sin(n\epsilon)$, cn stands for $\cos(n\epsilon)$ and $s_{1/2}$ for $\sin(\epsilon/2)$. The above is a pair of coupled equations and it was Aryabhata's insight to take the second difference, namely;

$$\delta 2sn = \delta sn - \delta sn-1 = 2s_{1/2}(c_{(n-1/2)} - c_{(n-3/2)})$$

$$= -4s^2_{1/2sn-1} \text{ on using Eq. (4)} \qquad (5)$$

Thus, the second difference of the sines is proportional to the sine itself. The next step is to represent the R.H.S in terms of a recursion. We observe sn on the R.H.S. of Eq. (5) may be written as $sn = sn_{\_}\, sn_{-1} + sn_{-1} - sn_{-2} + sn_{-2} - \ldots = \boldsymbol{\delta s}_{n-1} + \delta s_{n-2} + \ldots$ Thus,

$$\delta s_n - \delta s_{n-1} = -4s^2_{1/2}\sum_{m=1}^{n-1} \delta s_m$$

Thus, we get a recursion relation where the second difference of the sines is expressed in terms of all previously obtained first sine differences. To initiate the recursion we need δs_1 which is $s_1 - s_0 = \sin(\epsilon) - \sin(0) \approx \epsilon$ since for small angles the half-chord and the corresponding arc are equal as stated in the previous section.

Using the recursion relation of the sine, we can generate the celebrated sine table of Aryabhata, taking $\pi = 3.1416$ and $\sin(\epsilon) = \epsilon = 0.0654$ (= 225').

Table 2 depicts some typical values of the sine function as well as the value of the sine multiplied by 3438 (the so called *Rsine of Aryabhata*). We can see that this matches Aryabhata's celebrated sine table, up to 1 minute. For example, $\theta = \pi/6$ gives 1719 minutes. For comparison, we also showed the modern values of sine up to four decimal places. Note that Aryabhata takes angles up to $\pi/2$ and seemed aware of the fact that going further was unnecessary given the periodic nature of the sine function.

θ	sin(θ) Aryabhata	sin(θ) (Minutes)	sin(θ) Modern
$\pi/48$	0.0654	225	0.0654
$2\pi/48$	0.1305	449	0.1305
$3\pi/48$	0.1951	671	0.1951
$4\pi/48$	0.2588	890	0.2588

$5\pi/48$	0.3214	1105	0.3214
$6\pi/48$	0.3827	1315	0.3827
$7\pi/48$	0.4423	1520	0.4423
$8\pi/48$	0.5000	1719	0.5000
$9\pi/48$	0.5556	1910	0.5556
$10\pi/48$	0.6088	2093	0.6088
$11\pi/48$	0.6594	2267	0.6593
$12\pi/48$	0.7072	2431	0.7071
$13\pi/48$	0.7519	2585	0.7518
$14\pi/48$	0.7935	2728	0.7934
$15\pi/48$	0.8316	2859	0.8315
$16\pi/48$	0.8662	2978	0.8660
$17\pi/48$	0.8971	3084	0.8969
$18\pi/48$	0.9241	3177	0.9239
$19\pi/48$	0.9472	3256	0.9469
$20\pi/48$	0.9662	3322	0.9659
$21\pi/48$	0.9812	3373	0.9808
$22\pi/48$	0.9919	3410	0.9914
$23\pi/48$	0.9983	3432	0.9979
$24\pi/48$	1.0005	3439	1.0000

Table 2: Table of values of sine using the Aryabhatan method, taking episilon = $\pi/48$ (= 3.75 = 225′) and pi = 3.1416, and comparison with the modern-day values. In column 3, we also quoted values in minutes as done in Verse 10, Gitika chapter of the Aryabhatiya [1,2].

Finite Difference Calculus

Of greater relevance is the fact that the sine (or cosine) difference formulae foreshadow finite difference calculus, a popular numerical technique in this age of computation.

Rewriting Eqs. (1) and (2) with $\varphi = \epsilon$,

$$\frac{\sin(\theta + \epsilon) - \sin(\theta - \epsilon)}{2\sin(\epsilon)} = \cos(\theta) \quad (7)$$

$$\frac{\cos(\theta + \epsilon) - \cos(\theta - \epsilon)}{2\sin(\epsilon)} = -\sin(\theta) \quad (8)$$

Aryabhata took ϵ to be $\pi/48$. But he also stated that its value is yateshtani or as per our wish (Verse 11, *Ganitapada*). Some took it to be $\pi/96$ and others like Brahmagupta took it as $\pi/12$ or 15 degrees. If we take ϵ to be sufficiently small we have our classic formula for finite difference calculus. Noting that 2 sin(ϵ/2) ≈ ϵ we have the finite difference version of the derivative of sine:

$$\frac{\delta \sin(\theta)}{\delta\theta} = \cos(\theta)$$

and similarly for the cosine.

$$\frac{\delta \sin(\theta)}{\delta\theta} = -\sin(\theta)$$

Let us understand this with an example. We know that sine(37°) is close to 0.6 and sine (300) is 0.5. The difference in angle is 7° which in radians is 0.122. Thus, the derivative of sine of the median angle 33.5° from Eq. (7) is:

$\delta\sin(\theta)/\delta\theta = (0.6 - 0.5)/0.122 = .82$

Looking up the sine table or the calculator yields cos (33.5) = 0.83. Similarly Eq. (5) yields the second derivatives namely:

$\delta 2\sin(\theta)/\delta 2\theta \approx -\sin(\theta)$

$\delta 2\cos(\theta)/\delta 2\theta \approx -\cos(\theta)$

The above are now called central difference approximations to the derivative and the second derivative. Aryabhata does not mention the term finite difference calculus (let alone calculus). But similar methods are now used to numerically solve our differential equations. A student can readily recognise the above as a standard solution of the classical simple harmonic oscillator. Note also that Newton's II Law and the famous Schrodinger equation of quantum mechanics are both second order differential equations.

Discussion

One can discern a continuity in Indian mathematics, howsoever tenuous, from pre-Vedic times (← 1000 BCE) up and until 1800s. The Sulbasutras (circa 1000 BCE) reveal intricate geometric thinking. Baudhayana foreshadowed Pythagoras by a few centuries. Recursive thinking whose examples are evidenced in equations (3 to 5) has its roots in Pingala's Combinatorics (300 BCE). Aryabhata was an inheritor of this tradition. Equally relevant is the influence the Aryabhatiya exercised on Indian mathematicians. Bhaskara I (600 CE) wrote a commentary on it. Brahmagupta (600 CE) was influenced by it. Bhaskara II (1150 CE) used the trigonometric difference formulae to obtain the area and the volume of the sphere. Our presentation here is based on the voluminous work (Maha Bhashya) of the fifteenth century mathematician Nilakantha Somaiyaji (1444 CE – 1544 CE) who was part of the Kerala school which, beginning with Madhava (1350 CE – 1420 CE), founded the calculus of trigonometric functions. Aryabhata serves as a crucial link in Indian mathematics [4,5].

The sine table can also be generated using the half-angle formula. This was demonstrated in the *Panchasiddhantika* a text written barely 50 years after the appearance of the *Aryabhatiya* [7]. As pointed out, a feature of the Aryabhata's difference relation is how contemporaneous it is. It can be easily recognised as finite difference calculus. This perhaps led to the development of the calculus of trigonometric functions by the Madhava (1350 CE) and his disciples along the banks of the Nila river in Kerala. This school is variously called the Nila [4] and even as the Aryabhata school [6]. Another aspect to note is that Bhaskara II (1100 CE) used the canonical 2 π/96 division of the great circle to carry out discrete integration and obtain the (correct) expressions for the surface area and volume of the sphere. Jyesthdeva of the Nila (or Aryabhta) school in his work Yuktibhasa derived the same results using calculus (circa 1500 CE). Aryabhata can legitimately be called the founder of trigonometry.

To sum up, the *Arybhatiya* exercised a tremendous influence over Indian mathematicians and for over a thousand years. For a book with just over 100 pithy (short) verses, its legacy remains unparalleled in the scientific world. We hope that our article will give our young audience an introduction to his work and will serve as an inspiration.

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Exercises

1. We can generate the sine table as per Aryabhata's suggestion but not exactly using the same value for ϵ he used. We choose $\epsilon = \pi/80.93$ which is the same as 2.250. We take $\sin(\epsilon) = \epsilon$. If you have a simple calculator generate all values of sine from 2.25 to 18 degrees in equal steps using Eq. (6). Alternatively, if you have a programmable calculator or a computer generate all values of sine from 2.25 to 90 degrees. Compare with the results your calculator will otherwise yield.
2. Show that the two triangles BSC and OPA in Fig. 2 are similar.
3. In the last section reference is made of the text Panchasiddhantika wherein the half-angle forumla is mentioned, *viz*.

$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

How would you (i) derive this by a geometrical construction; (i) employ this to generate the sine table?

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This work is in Sanskrit and there is no English (or Hindi) translation of this seminal text to the best of our knowledge. The other works mentioned herein are in English.