SNIPPETS OF ANCIENT INDIAN MATHEMATICS

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It is important that mathematics is taught with a sense of its history. It is not as though one needs to study the history of mathematics as a separate subject. Rather, it is appropriate to bring in problems and contexts from history into the study of mathematical topics wherever feasible and relevant. This can add immensely to the appreciation and enjoyment of the subject. This article is based on a talk given on 11 November, 2022 for the "Listening to Learn" Webinar series of the National Council Educational Research and Training (NCERT). In the talk, we dealt with four problems, each of which has a connection with some topic from ancient Indian mathematics: (a) Magic squares of orders 3 and 4; (b) The mathematics of compositions and prosody, and the Fibonacci numbers, also known as the Virahanka-Gopala-Hemachandra numbers; (c) Aryabhata's kuttaka ('pulverizer') algorithm and the jugs-and-water-cups problem; (d) Brahmagupta triangles. The focus in each case was on the mathematics of the problem and not on the history; but we also touched upon the historical aspect. We have followed the same style in this article. There is great beauty and richness in studying such problems. It would have great value if high school mathematics were taught with glimpses of such gems.

Keywords: Magic square, pan-diagonal magic square, Virahanka-Fibonacci numbers, compositions, recursion, Kuttaka algorithm, Bhavana, Brahmagupta triangle

Magic Squares of Order 3

Magic squares have been known since ancient times and continue to be a source of great pleasure.

We begin by talking about magic squares of order 3. To start, we pose the following question:

In a magic square using the numbers 1, 2, 3, ..., 8, 9, what number must occupy the central square?

The wording of the question seems to suggest that there is only one possible answer. This is so. Here is an engaging and eye-opening proof of this claim found 'live' during a math club session with middle school students.

A Surprising Solution

Since 1+2+3+ ... +9 = 45, the magic sum is 45/3 = 15. Let be the central number; then is one of the numbers . Figure 1 shows what the configuration looks like:

	m	

Fig. 1: Finding the central number of a third order Magic Square.

We now consider the different possibilities:

- Suppose m = 9. Then in the ring around the central cell, we must find the number 8 somewhere. But 9 + 8 = 17 > 15 is already too large; the third number in the line corresponding to 9 and 8 would have to be -2 which is not permitted! Hence, this configuration cannot be successfully completed into a magic square. Therefore, the possibility $m = 9$ is ruled out.

- The same reasoning works for the possibilities $m = 8, 7, 6$. It follows that $m \neq 9, 8, 7, 6$.
- We now consider the remaining possibilities.
- Suppose $m = 1$. Then in the ring around the central cell, we must find the number 2 somewhere. But as 1 + 2 = 3, the third number in the line corresponding to 1 and 2 would have to be 12, which is too large! Hence, this configuration cannot be successfully completed into a magic square. Therefore, the possibility is ruled out.
- The same reasoning works for the possibilities $m = 2, 3, 4$. It follows that $m \neq 1, 2, 3, 4$.
- This means that $m \neq 9, 8, 7, 6$ and $m \neq 1, 2, 3, 4$.
- There is just one possibility left for the number in the central square! So, we have $m = 5$.

Armed with the knowledge that the magic sum is and the central number is , it is easy to construct the magic square:

8	1	6
3	5	7
4	9	2

Fig.2: The completed 3rd order Magic Square.

Another way of finding the central number

We now show a very different approach here to the same problem. It is worthy of close study.

Draw lines through the central square to cover all the squares (see Figure 3). Note that 4 such lines are needed. The sum of the numbers on each line is 15 (this being the magic sum of the square). Therefore, the total of these four sums is $4 \times 15 = 60$.

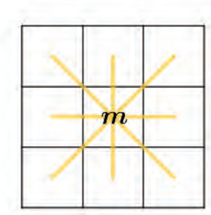

Fig.3: Another approach for finding the central number.

Now, observe that the central number is covered 4 times, as it lies on each line; the other numbers are covered exactly once each. This implies that

$60 = (1 + 2 + 3 + ... + 9) + 3m = 45 + 3m$

giving $m = 5$, we have obtained the same answer as earlier.

Pedagogic points to be kept in mind by the teacher

1. It is not possible to have a magic square of order two. Students should be invited to demonstrate this fact.
2. It is important that students get to see both the above proofs as they illustrate different kinds of reasoning and different approaches to proof:

   a. The first of these is proof by exhaustion (also called brute force enumeration). In some settings this may well be the only approach available.

b. The second is more sophisticated as it depends on the use of algebra and equations, which have first to be set up.

3. There are four different arithmetic progressions in the magic square. This may provide a nice entry point into the study of APs. The topic enters in a very natural manner.

4. We noted above, after showing that the central number is 5, that the remaining cells are easy to fill in. At various points we are confronted with choices, but the different choices only lead to different orientations of the same basic design. This means that there is 'essentially just one' magic square of order three using the numbers from 1 to 9. The use of the phrase 'essentially just one' immediately points to the idea of symmetry. Therefore, playing with this most basic of magic squares introduces us to ideas of symmetry in a very natural manner.

5. Is it possible to use the basic design of the magic square of order three to make a magic square of order six, using the numbers from 1 to 36? This exploration holds great promise!

6. Is it possible to play around with the definition of a magic square and define the notion of a magic rectangle? This notion would be slightly more general than that of a magic square.

Magic Squares of Order 4

Next, we study magic squares of orders 4. Here the problem is that of arranging the numbers from 1 to 16 in a 4×4 square array so that the four rows, the four columns, and the two main diagonals, all have the same magic sum. Since the sum of the numbers from 1 to 16 is $\frac{16 \times 17}{2} = 8 \times 17 = 156$, it follows that the magic sum of such a square is $2 \times 17 = 34$.

Unlike the situation for the 3×3 magic square, where using the numbers 1, 2, ..., 8, 9 there is essentially just one design for the magic square, here we find many different designs possible. Indeed, it is a non-trivial problem to find the number of different designs.

We now look at a pair of fourth order magic squares associated with two famous historical artefacts (Figure 4).

Let us focus our attention on the magic squares in these two artefacts (Figure 5).

In Figure 6 we see the two magic squares displayed in plain text form.

Now, it turns out that all fourth order magic squares share a certain non-obvious and non-trivial property illustrated in Figure 7. But it is quite tricky to prove!

Albrecht Durer's Melencolia

Parshvanatha temple, Khajuraho

Fig. 4: Albrecht Dürer's Melencolia and Parshvanatha temple, Khajuraho. Source: [1]

Dürer's magic square

Khajuraho magic square

Fig. 5: Close ups of the magic squares. Source: [1]

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig. 6: The two magic squares shown in plain text.

a			b
	c	d	

Fig. 7: For a 4 × 4 magic square, if *a*, *b*, *c*, *d* are as shown, then $a + b = c + d$. Moreover, this is true for each pair of symmetrically placed rows, columns, and diagonals!

Do these two fourth order magic squares have the property pointed out in Figure 7? Yes, they do!

For example, in Dürer's magic square we have:

$$16 + 13 = 15 + 14$$
$$5 + 9 = 13 + 1$$
$$5 + 8 = 6 + 7$$
$$6 + 11 = 16 + 1$$

And in the Khajuraho magic square we have:

$$7 + 14 = 6 + 15$$
$$2 + 16 = 14 + 4$$
$$2 + 11 = 3 + 10$$
$$3 + 8 = 7 + 4$$

But the Khajuraho magic square has additional properties — it is truly magical! We focus on one set of such properties: the pan-diagonal nature of the square. We illustrate the meaning of this in the displays shown below. Both the displays show the Khajuraho magic square, but with different numbers highlighted. In both cases, the highlighted numbers form a "broken diagonal'" — i.e., a diagonal with a 'wraparound' effect.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig. 8: The 'broken diagonals' add up to the magic sum.

Next, note the properties of the 2 × 2 sub-squares. For example, at the top left, we have a 2 × 2 sub-square with entries 7, 12, 13, 2:; the sum of these numbers is 34 which is the magic sum of the square. (See Figure 9.) This property holds for all the 2 × 2 sub-squares!

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig. 9: The entries in each 2×2 sub-square add up to the magic sum.

Similarly, note the sub-squares. We find that for every such sub-square, the sum of the numbers at the corners of the square is half the magic sum of the square. (See Figure 10.)

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig. 10: The corner entries in each 3 × 3 sub-square add up to half the magic sum.

For example, at the top left we have a 3 × 3 sub-square with corner entries 7, 1, 10, 16; observe the equalities 7 + 10 = 17 = 1 + 6. This property holds for every single sub-square!

The Khajuraho magic square is truly magical!

Who is the discoverer of this remarkable square? It is the mathematician Narayana Pandit of the 14th century. We list below some of his pioneering work on magic squares.

- His text Ganita Kaumudi describes how to construct magic squares of different orders. It also describes how to construct all possible pan-diagonal magic squares of order four. He proves that the total number of possible pan-diagonal magic squares of order four is 16 × 24 = 384. This is a non-trivial result! He also shows how to construct shapes such as circles, rectangles, and hexagons with similar properties.
- Very remarkably, he states that the purpose of studying this topic is "...to destroy the egos of bad mathematicians, and for the pleasure of good mathematicians."

An interesting insight comes when we realise that in medieval times, magic squares were truly regarded as magical. For example, in Narayana Pandit's text he states that a magic square with sum 20 is useful in cases of poisoning; a magic square with sum 28 is useful when one's paddy field is attacked by insects; and a magic square with sum 84 is useful to quieten children when they are crying.

Another insight into the nature of the Indian mind comes when we observe that combinatorial designs were of great interest in ancient India. The ancient Greeks, in contrast, were much more interested in pure geometry. (A common area of interest in both cultures was number theory).

In terms of pedagogy, the study of pan-diagonal magic squares holds the following significance:

It yields an entry point into the study of symmetry of different kinds.

It demonstrates that "trial-and-error" and experimentation are essential parts of mathematics.

The wraparound property of a pan-diagonal magic square yields a nice introduction to the study of an object like a torus.

The Virahanka-Gopala-Hemachandra-Fibonacci Sequence

The Fibonacci sequence is generally defined using an arithmetic rule: the specification that each number in the sequence after the first two is the sum of the previous two numbers. Starting with the numbers 0 and 1, we obtain the following sequence:

0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597, ...

But these numbers can also be defined combinatorially. We now show how.

But first, we study another such problem: counting the compositions of the positive integers.

Compositions of the positive integers

A composition of a positive integer is an expression for as an ordered sum of positive integers; if we change the order of the summands, we get a different composition. The compositions of 3 are the following: 3; 2 + 1; and 1 + 2; and 1 + 1 +1. Let a_n denote the number of compositions of n; The above listing tells us that $a_1 = 1$ and $a_2 = 1$

1. We may generate more values through experimentation.

Problem: Find a formula for a_n .

n	Compositions of n	a_n
1	1	1
2	2; 1 + 1	2
3	3; 2 + 1; 1 + 2; 1 + 1 + 1	4
4	4; 3+1; 1+3; 2+2; 2+1+1; 1+2+1; 1+1+2; 1+1+1+1	8
5	5; 4+1; 1+4; 3+2; 2+3; 3+1+1; 1+3+1;1+1+3;2+2+1;2+1+2;1+2+2;2+1+1+1; 1+2+1; 1+1+2+1; ;1+1+1+2; 1+1+1+1	16

Examine the third column. What a lovely pattern! It invites us to find an equally lovely proof.

Conjecture: $a_n = 2^{n-1}$.

A proof using recursion

Observe that there is just one composition of having only one summand, namely: n itself. Let us set this composition aside and focus on the compositions with more than one summand.

Take any composition with more than one summand; let its first term be k, where $1 \leq k \leq n-1$. So:

Here, note that the bracketed term (a+b+c+ ...) forms a composition of $n - k$. If we keep the first term fixed at k, we can complete the composition with any composition of $n-k$. It follows that there are a_{n-k} compositions in which the first term is k.

Now, sum this result over all values of k. Bringing back the composition with a single summand, we see that

$$a_n = 1 + a_1 + a_2 + ...+ a_{n-2} + a_{n-1}$$

Replacing n by $n-1$, we get

$$a_n = 1 + a_1 + a_2 + ...+ a_{n-3} + a_{n-2}$$

Examining these two relations, we conclude that $a_n = a_{n-1} + a_{n-1}$ i.e,

$$a_n = 2a_{n-1}$$

It follows that a_1, a_2, a_3 is a doubling sequence! Since $a_1 = 1 = 2°$ we conclude that $a_n = 2^{n-1}$. We have proved the conjecture.

Mathematics of Poetry

In poetry, the term prosody refers to 'rhythm, intonation and speech'. We now study a problem whose origins lie in prosody. It was first studied by grammarians such as Pingala, in the second century BCE, and mathematicians such as Virahanka (700 CE), Gopala (1135 CE), and Hemachandra (1150 CE).

We start by classifying the syllables of the language into two categories: light and heavy. Light syllables (L) have weight 1, while heavy syllables (H) have weight 2. Words can be formed using both kinds of syllables.

There is clearly just one word possible with weight 1, namely: S. And there are just two words possible with weight 2: H and LL. The situation invites us to formulate the following problem:

Problem. Find the number of words with total weight n.

Let this number be c_n. Like earlier, we generate the relevant data by experimentation:

n	Words with total weight n	C_n
1	*L*	1
2	*H;LL*	2
3	*HL; LH; LLL*	3
4	*HH; HLL; LHL; LLH; LLL*	5
5	*HHL; HLH; LHH; HLLL; LHLL; LLHL; LLLH; LLLLL*	8
6	*HHH; HHLL; HLLH; HLHL; LHHL; LLHH; LHLH; HLLL; LLHLL*	13

Here is a table of values of c_n:

n	1	2	3	4	5	6
C_n	1	2	3	5	8	13

Observation: c_1, c_2, c_3 is a (displaced) Fibonacci sequence!

But how do the Fibonacci numbers enter the topic of prosody? It turns out that we can use an argument exactly like the one we used to count compositions.

- Consider all words with total weight where $n > 2$.
- Each of these must end L or H. Delete the last syllable from each word.
- If the deleted letter is L, then the reduced word has weight $n - 1$.
- If the deleted letter is H, then the reduced word has weight $n - 2$.
- In each case, complete families are obtained.

For example, $n = 5$. As the above table indicates, there are 8 words with total weight 5:

HHL; HLH; LHH; HLLL; LHLL; LLHL; LLLH; LLLLL

On deleting the last syllable of each word and separating the remaining words into two classes, depending upon whether the deleted letter was L or H, we obtain the following picture:

HH; HLL; LHL; LLH; LLL	*HL; LH; LLL*
Words where the deleted letter was L. All these words have total weight 4. There are five words in this collection, corresponding to $c_4 = 5$.	Words where the deleted letter was H. All these words have total weight 3. There are three words in this collection, corresponding to $c_3 = 3$.

It is easy to see from the argument developed above that c_5 must be equal to $c_4 + c_3$.

Though we have sketched the argument for the case $n = 5$, the argument is perfectly general, and it proves that

$$c_n = c_{n-1} + c_{n-2} \textit{ for all } n > 2$$

In other words, the -sequence follows the Fibonacci recurrence.

At the same time, $c_1 = 1$ and $c_2 = 2$. So, the -sequence starts with a pair of consecutive Fibonacci numbers.

These conditions suffice to imply that every c-number is a Fibonacci number. It follows that the -sequence is a displaced Fibonacci sequence. Indeed, $c_n = F_{n+1}$ where F_k denotes the k-th Fibonacci number.

Comment: The Fibonacci numbers are also known as the *Virahanka-Gopala-Hemachandra numbers*, because the Indian mathematicians Virahanka, Gopala and Hemachandra discovered them much before Fibonacci. But they discovered these numbers using combinatorial reasoning. See [5].

Pedagogical Notes

Students going through this exploration, experience many things that are important from a pedagogical perspective:

- Generating a sequence empirically.
- Organising data systematically and efficiently.
- Spotting patterns in data and formulating a suitable conjecture.
- Proving the conjecture using recursion. Or disproving it by checking with more data.
- Formulating variations of the same basic problem.

Aryabhata and the Jug Problem

In the 5th century Aryabhata described an algorithm to solve linear Diophantine equations in a single variable. (The word 'Diophantine' tells us that we are interested only in integer solutions). He called this the kuttaka or the 'pulveriser.' See [6] for details.

Fig.11: displays the relevant verse from Aryabhata's famous text.

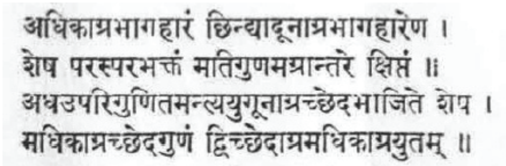
अधिकाग्रभागहारं छिन्द्यादूनाग्रभागहारेण ।
शेष परस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तं ॥
अधउपरिगुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेष ।
माधिकाग्रच्छेदगुणं द्विच्छेदाग्रमधिकाग्रयुतम् ॥

Fig. 11: A verse from Aryabhatiya. Source: [6]

On examination we find that the algorithm is largely like Euclid's division algorithm for the determination of the greatest common divisor (GCD) of two given positive integers. Euclid's algorithm may be described as follows. Given two positive integers a,b where $a > b$, we replace the ordered pair (a,b) by

$$(b,r)$$

where is the remainder $a \div b$. Since $r < b$, the numbers in the new pair are strictly smaller than those in the original pair. We now iterate this operation till it concludes (which it must). Aryabhata's algorithm is similar but uses subtraction rather than division. Its purpose is more general. Given two positive integers with GCD g, it seeks to find a pair of integers x, y such that

$$ax + by = g.$$

Note that if $(x,y) = (u,v)$ is a solution then so is $(x,y) = (u - nb, v + na)$ for any integer. The algorithm yields an algorithmic proof to the following theorem:

Given two positive integers a,b with GCD g, integers x,y can be found such that $ax + by = g$. If a, b are co-prime then integers x,y can be found such that $ax + by = 1$.

Our focus now will not be to dwell on the algorithm as such but to show an entertaining and unexpected connection between the solution offered by Aryabhata's algorithm and a well-known puzzle involving jugs and tumblers and a tank of water. Movie buffs will be interested to know that this puzzle played a part in one of the Die-Hard movies!

The puzzle may be stated as follows. Say we are given two cups with capacities a litres and b litres, and a tank with a large amount of water. Here, a, b are given positive integers. Using these two cups we must, through a series of exchanges (i.e., filling the cups and emptying them) end up with exactly 1 litre in one cup. Estimation is not permitted; we are not allowed to fill up or empty either cup by some fractional amount. The desired result is achievable if and only if a,b are coprime. Given that this condition holds, we now show how the *kuttaka* algorithm can indicate the steps needed. We start by finding a pair of integers such that $au + bv = 1$. The actions to be performed are now dictated by the values of u,v.

Example

Suppose that (a,b) = (7,5). These are co-prime integers. We easily verify that with (u,v) = (-2,3) we get $au + bv$ (i.e., ((-2) × 7) + (3 × 5) = 1. This tells us that the 5-litre cup needs to be filled up 3 times, and the 7-litre cup needs to be emptied 2 times. The actions to be performed are the following:

Step	Action to be performed	Amount in 5-litre cup	Amount in 7-litre cup
1	Fill the 5-litre cup from the tank	5	0
2	Empty contents of 5-litre cup into 7-litre cup	0	5
3	Fill the 5-litre cup from the tank	5	5
4	Empty contents of 5-litre cup into 7-litre cup	3	7

5	Empty contents of 7-litre cup into the tank	3	0
6	Empty contents of 5-litre cup into 7-litre cup	0	3
7	Fill the 5-litre cup from the tank	5	3
8	Empty contents of 5-litre cup into 7-litre cup	1	7
9	Empty contents of 7-litre cup into the tank	1	0

Observe the following:

The 5-litre cup has been filled from the tank on three occasions: steps 1, 3, and 7.

The 7-litre cup has been emptied into the tank on two occasions: steps 5 and 9.

The figures agree with the relation ((-2) × 7) + (3 × 5). (The result may be achieved in different ways; the above sequence of actions is not the only one possible).

Before closing, we remark that Aryabhata's interest in solving linear indeterminate Diophantine equations come from astronomy: the problem of finding instances of occultation among the planetary bodies. This is of interest to astronomers and astrologers alike!

Brahmagupta Triangles

Consider a triangle with sides 3,4,5. Observe that:

Its sides are consecutive positive integers.

It has integer area; for, it is right-angled (with legs 3,4), so its area is $\frac{16\times17}{2} = 6$.

A triangle having all these specifications is called a Brahmagupta Triangle. If we only insist that the sides are integers (not necessarily consecutive integers), then the triangle is called a *Heron Triangle*.

Is it possible for us to find all possible Brahmagupta triangles? Are there infinitely many of them?

In the 7th century CE, Brahmagupta found many such triangles, with sides (3,4,5), (13,14,15), (51,52,53), (193,194,195), ... Looking carefully at his working, it seems clear that he knew how to generate infinitely many solutions to the problem. Let us see if we can unravel his approach.

Consider a triangle with sides $a-1$, a, a, where $a \geq 3$ is a positive integer. Its semi-perimeter is $\frac{3a}{2}$, so its area b is given by

$$b^2 = \frac{3a}{2}\left(\frac{a}{2}-1\right)\frac{a}{2}+1)$$

Hence:

$$16b^2 = a^2 \cdot 3(a^2-4)$$

Since $16b^2$ and a^2 are perfect squares, $3(a^2-4)$ is a perfect square as well, implying that

$$a^2-4 = 3c^2$$

for some integer . Therefore, to enumerate Brahmagupta triangles, we need to solve the equation

$$a^2 - 3c^2 = 4$$

over the set of positive integers. To get a sense of the family of solutions of this equation, let us generate some solutions using computer software:

n	1	2	3	4	5	...
a_n	4	14	52	194	724	...
c_n	2	8	30	112	418	

We immediately spot a striking pattern: both sequences show an identical two-term recurrence. Namely:

$$a_n = 4a_{n-1} - a_{n-2}$$
$$c_n = 4c_{n-1} - c_{n-2}$$

Only the initial terms are different: $a_1 = 4$, $a_2 = 14$, while $c_1 = 2$, $c_2 = 8$.

It is easy to show, using the principle of induction, that if the following are Brahmagupta triangles,

$$(a_{n-2} - 1, a_{n-1}, a_{n-2} + 1),$$
$$(a_{n-1} - 1, a_{n-1}, a_{n-1} + 1),$$

then so is $a_n - 1, a_n, a_{n+1}$ where is given by the recurrence relation shown above.

How do we explain this recurrence relation? To do so, we need to look more closely at the equation.

Analysis of the equation $a^2 - 3c^2 = 4$

If c is odd, then c^2 is of the form 1 (mod 4) which leads to $a^2 \equiv 3$ (mod 4). But no perfect square is of the form 3 (mod 4). As this possibility does not work out, c is even. Therefore, a too is even. Let $a = 2x$ and $c = 2y$ where x, y are positive integers. Substituting in the above equation and simplifying the expressions, we get

$$x^2 - 3y^2 = 1$$

This is an instance of a Brahmagupta-Pell equation. Such equations were first studied by Brahmagupta in the 7th century. Much later they were studied by European mathematicians such as Fermat and Euler (who mistakenly attributed some results to a British mathematician named Pell; the name has stuck ever since).

The 'Bhavana' Operation

Brahmagupta discovered an extraordinary 'law of composition' governing the set of solutions of such equations. He found that if (a,b) and (c,d) are solutions of $x^2 - 3y^2 = 1$, then so is the pair $(ac + 3bd, ad + bc)$. We may thus write:

$$(a,b) \otimes (c,d) = (ac + 3bd, ad + bc)$$

He named this operation Bhavana. A law of composition of this kind is familiar to us from a modern standpoint. But Brahmagupta was the first mathematician to study such ways of composing elements of a set. Using this law, he was able to generate any number of solutions to the given equation!

Examples of Bhavana

Let us compose the solution (2, 1) with itself. We get:

$$(2, 1) \otimes (2, 1) = (4 + 3, 2 + 2) = (7, 4).$$

Recalling that a = 2x and c = 2y, we get the Brahmagupta triangle (13, 14, 15).

Let us compose the solutions (7,4) and (2,1); we get:

$$(7, 4) \otimes (2, 1) = (14 + 12, 7 + 8) = (26, 15).$$

From this solution we get the Brahmagupta triangle (51, 52, 53) .

It is easy to see once we have the law of composition in our possession, that we can generate infinitely many solutions to the given equation.

A Jewel of Ancient Indian Mathematics

Brahmagupta's law of composition is one of the jewels of ancient Indian mathematics. In later centuries, the idea was developed further by Bhaskara II (who wrote the Lilavati). And in recent years, Manjul Bhargava has published some beautiful results related to the Bhavana.

In conclusion, we ask ...

What is the value of studying topics from our ancient past? Does it have any significance?

To answer that, we quote a famous poem from John Keats.

A thing of beauty is a joy for ever:

Its loveliness increases; it will never

Pass into nothingness ...

That brings me to the end of the talk. I have shared with you some gems from ancient Indian mathematics. I hope we have been able to experience their beauty together!

Declaration

The author affirms that there has been no conflict of interest in the writing of this paper.