REFLECTION SYMMETRY IN PHYSICS

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Arguments based on symmetry are quite common in Physics. Usually at the higher secondary stage, these involve spherical and axial symmetry. In this article, we discuss another interesting symmetry, the space reflection symmetry, that is less emphasised but can be easily incorporated in the school curriculum at that stage.

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Introduction

Arguments based on symmetry (spherical and axial) are commonly used in higher secondary physics. For example, a standard problem treated in NCERT, Class XII textbook is the calculation of electric field $\vec{E}$ outside and inside a uniformly charged spherical shell of total charge *Q* and radius *R*. We expect on the basis of spherical symmetry that the field is radial in direction (i.e., along the line from the centre of the shell *O* to the observation point *P*). Further, its magnitude can depend only on the radial co-ordinate r, but not on the angular co-ordinates θ, ϕ. Here, *r*, θ, ϕ are the usual spherical polar co-ordinates.

This is so since for the given problem, there is no privileged direction — all directions are on the same footing. By choosing a spherical Gaussian surface of radius r concentric with the shell, Gauss's law then gives the magnitude of electric field *E* to be $\frac{|Q|}{4\pi \epsilon_o r^2}$ if $r > R$ (outside the shell) and zero if $r < R$ (inside the shell), as given in the text. At the shell itself $r = R$, there is a discontinuity at every point equal to $\frac{|Q|}{4\pi \epsilon_o R^2} = \frac{|\sigma|}{\epsilon_o}$, where σ is the uniform surface charge density of the shell.

In the same way, to calculate $\vec{E}$ outside a long uniformly charged thin wire or the magnetic field $\vec{B}$ due to a long wire carrying a steady current, the use of axial (or cylindrical) symmetry is useful.

The electric field due to an infinite uniformly charged plane, however, involves a different symmetry (plane symmetry). Axial symmetry involves symmetry about a given direction fixed in the problem. In the case of the infinite uniformly charged plane, that axis can be any normal to the plane. This means all points and directions on the plane are on the same footing and the field must be normal to the plane at every point. The field in this case is found to be uniform throughout with magnitude $\frac{|\sigma|}{2\epsilon_o}$ but directed outward (inward) normal on either side for $\sigma > 0$ ($\sigma < 0$), resulting in a discontinuity in electric field of magnitude $\frac{|\sigma|}{\epsilon_o}$ at the plane, the same value as for the spherical shell.

This article describes another symmetry, called the space reflection or inversion symmetry, that is simple, useful and enriches our understanding of physics. We start with

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some basic definitions and comments, and then see what this symmetry implies in a few examples.

Scalars and Vectors: Behaviour Under Reflections

We know that the scalar or vector character of each term in any equation in physics should match. We cannot add a scalar to a vector, which means one side of an equation cannot be a scalar when the other side is a vector. For example, Newton's II law of motion relates force $\vec{F}$ on the particle to the rate of change of its momentum $\vec{p}$:

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad (1)$$

where each side is a vector. The II law leads to the Work Energy Theorem:

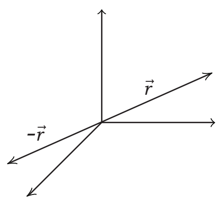
$$\vec{F}.d\vec{s} = dK; \; K = \frac{1}{2}mv^2 \qquad (2)$$

where each side is a scalar. The left side denotes work done by the force in the infinitesimal displacement $d\vec{s}$ and the right side is the infinitesimal change in kinetic energy *K*.

Let us now see the behaviour of quantities under reflection through the origin corresponding to the transformation:

$$\vec{r} \rightarrow -\vec{r} \qquad \text{Reflection (3)}$$

Here $\vec{r}$ is the position vector of a particle relative to some origin. This transformation is equivalent to changing a right-handed co-ordinate system to a left-handed one.



A. Polar and Axial Vectors

There are two types of vectors depending on how they behave under reflection. Vectors that change sign under reflection are called polar vectors. Vectors that do not change sign under reflection are called pseudo-vectors (also called axial vectors). From the definitions of the velocity $\vec{v}$, and momentum $\vec{p}$, they are obviously, like $\vec{r}$, polar vectors.

$$\vec{v} \rightarrow -\vec{v} \;\; ; \;\; \vec{p} \rightarrow -\vec{p} \qquad (4)$$

From the II law or the usual examples of forces, such as the gravitation force or Coulomb force, we infer that the force $\vec{F}$ is also:

$$\vec{F} \rightarrow -\vec{F} \qquad (5)$$

Next, see the vector product of two such vectors. Consider the vectors, orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ and torque $\vec{\tau} = \vec{r} \times \vec{F}$. Clearly, under reflection:

$$\vec{L} \rightarrow +\vec{L}; \vec{\tau} \rightarrow +\vec{\tau} \qquad (6)$$

Thus, angular momentum $\vec{L}$ and torque $\vec{\tau}$ are pseudo-vectors. Let us next consider electric and magnetic quantities. Coulomb's law for electric field $\vec{E}$ due to a charge *Q* is:

$$\vec{E} = \frac{Q}{4\pi \epsilon_0 r^3}\vec{r} \qquad (7)$$

Thus, $\vec{E}$ is a polar vector. Next, the Biot-Savart law for magnetic field $\vec{B}$ due to a line element $\vec{dl}$ of a wire carrying a steady current I is given by,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{l} \times \vec{r}}{r^3} \qquad (8)$$

Since the right-hand side is a vector product of two polar vectors, the line element $\vec{dl}$ and the position vector $\vec{r}$, $\vec{B}$ is pseudo-vector.

In a similar way, we can see how other electromagnetic quantities transform under reflection using their definitions. For example, electric dipole moment $\vec{\mu}$, the polarisation vector $\vec{P}$ and electric displacement vector $\vec{D}$ are all polar vectors. In contrast, magnetic dipole moment $\vec{m}$, magnetisation $\vec{M}$ and magnetic intensity $\vec{H}$ are pseudo-vectors.

B. Scalars and Pseudo-scalars

Here again, they are two types of scalars depending on what happens to their value under reflection. We refer to scalars which do not change sign under reflection as proper scalars; those that change sign under reflection are called pseudo-scalars. In this context, the term scalar is generally taken to imply proper scalar, not pseudo-scalar.

Consider $\vec{F} \cdot \vec{v}$. It is a scalar product of two polar vectors. Under reflection, since each changes sign, the product does not change sign. So $\vec{F} \cdot \vec{v}$, which physically means power delivered by the force, is a proper scalar. Next, take the scalar product of a polar vector $\vec{E}$ and the pseudo-vector $\vec{B}$: $\vec{E} \cdot \vec{B}$. Since under reflection $\vec{E}$ changes sign but $\vec{B}$ does not, the product changes sign under reflection; so $\vec{E} \cdot \vec{B}$ is a pseudo-scalar.

In general, the scalar product of two polar vectors or two axial vectors is a proper scalar, while the scalar product of a polar and axial vector is a pseudo-scalar. On the other hand, the vector product of two polar vectors or two pseudo-vectors is a pseudo-vector, while the vector product of a polar vector and a pseudo-vector is a polar vector. These statements follow easily from the earlier definitions.

Invariance of Laws Under Reflections

The laws of mechanics and electrodynamics are invariant i.e. they retain the same FORM under reflections. This is called the reflection symmetry of the laws. Note that the quantities appearing in a law may or may not change under reflection. Invariance of the law means the equation describing the law remains the same for the transformed quantities. Another way to state reflection symmetry is that the laws look the same for right-handed and left-handed co-ordinate systems.

What this means is that the L.H.S and R.H.S of every equation always have the same character (polar or pseudo-vector) or (scalar or pseudo-scalar). This is evident in mechanics in the basic laws:

$$\vec{F} = \frac{d\vec{p}}{dt} \; ; \; \vec{\tau} = \frac{d\vec{L}}{dt} \qquad (9)$$

Note that in the first equation of (9), both sides of the equation are polar vectors. In the second equation of (9), both are pseudo-vectors. Likewise, both sides of the Work Energy Theorem (2) are proper scalars.

In the same way, we can check that the laws of electrodynamics satisfy reflection symmetry. For example, the Lorentz force law on a charge q with velocity $\vec{v}$ in external electric and magnetic fields, given by,

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}) \qquad (10)$$

is invariant under reflection. To see this, note that the vector product of the polar vector $\vec{v}$ and the pseudo-vector $\vec{B}$ is a polar vector, so each term in (10) is a polar vector; hence, it retains its form under reflection. Another familiar formula is the potential energy *U* of an electric dipole placed in an external electric field or that of a magnetic dipole in an external magnetic field is:

$$U = -\vec{\mu}.\vec{E}\ ; \quad U = -\vec{m}.\vec{B} \qquad (11)$$

We know in mechanics, *U* is a proper scalar. The right side of first equation in (11) involves the scalar product of two polar vectors, which is a proper scalar. The second equation in (11) is a scalar product of two pseudo- vectors, which again is a proper scalar. The character of both sides thus, matches, showing that (11) is reflection invariant.

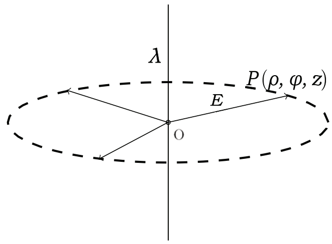
The fundamental laws of electrodynamics — the Maxwell's equations, are readily seen to be invariant under reflections. As they involve vector calculus, we will not write them down and show this explicity here.

Examples

A. Electric field outside an infinitely long uniformly charged wire

We wish to first find the direction of the electric field *E* at any point *P* in space, due to an infinitely long wire of uniform linear charge density λ.

Choose the origin *O* on the wire in the plane normal to the wire passing through *P*. The position vector of *P* is $\vec{\rho}$. There is no other relevant vector in the problem to determine the direction of $\vec{E}$. Both $\vec{E}$ and $\vec{\rho}$ are polar, so consistent with reflection symmetry, we can say that $\vec{E}$ must be along $\vec{\rho}$. That is $\vec{E} = E\hat{\rho}$, where $\hat{\rho}$ is a unit vector in the direction of $\vec{\rho}$.



What about the magnitude of $\vec{E}$? There is symmetry with respect to rotation around the wire taken to be the *z*-axis. This axial symmetry says that *E* cannot depend on ϕ. Further, since the wire is infinitely long, *E* cannot depend on *z* either. Thus, the symmetry arguments give:

$$\vec{E} = E(\rho)\hat{\rho} \qquad (12)$$

Reflection symmetry argument will not give you E(ρ) explicitly. Axial symmetry and use of Gauss's law on a cylindrical surface around the wire as its axis gives the standard textbook result:

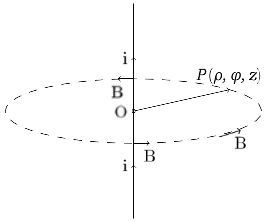
$$\vec{E} = \frac{\lambda}{2\pi \epsilon_o \rho}\hat{\rho} \qquad (13)$$

B. Magnetic field outside an infinitely long wire carrying steady current

We wish to first find the direction of magnetic field $\vec{B}$ at any point *P* in space, due to an infinitely long wire carrying a steady current *i*. Here, a reflection symmetry-based argument again helps.

In this case, there are two relevant vectors in the problem: the line element $\vec{dl}$ of the wire in the direction of current and the radial vector $\vec{\rho}$, both polar. The magnetic field $\vec{B}$ is a pseudo-vector. The only way to get a pseudo-vector from the two relevant polar vectors is from the vector product $\vec{dl} \times \vec{\rho}$.

Taking the positive z-axis to be along the wire in the direction of current, the direction of $\vec{dl} \times \vec{\rho}$ is the unit vector $\hat{\phi}$. Thus, $\vec{B} = B\hat{\phi}$.



As before, *B* cannot depend on ϕ due to axial symmetry and not on *z* either, since the wire is infinitely long. Thus,

$$\vec{B} = B(\rho)\hat{\phi} \tag{14}$$

Again, reflection symmetry argument cannot give you B(ρ) explicitly. Axial symmetry and Ampere's law applied to a circular loop centred at the wire passing through *P* gives the standard textbook result:

$$\vec{B} = \frac{\mu_o i}{2\pi\rho}\hat{\phi} \tag{15}$$

Under reflection $\vec{\rho}$ changes sign, and the direction of current reverses, but $\vec{B}$ does not change its direction, since, it is a pseudo-vector.

Violation of Reflection Symmetry in Nature

One basic point needs to be noted. A symmetry in a force law does not mean say the orbit of a particle governed by that force must also show that symmetry. For example, the gravitational force on a planet due to the Sun is an inverse square force depending only on the distance between the two. Taking the Sun at the origin, the force is spherically symmetric about the origin. This does not imply that the planar orbit of a planet must necessarily be circular. In general, a planetary orbit is elliptical.

The same is true for the Coulomb force on an electron in an atom due to the nucleus at the origin. The electron orbits in an atom are in general elliptical. In more accurate terms, the quantum atomic orbitals need not be spherically symmetric functions, even though the Coulomb potential is spherically symmetric.

Similar reasoning holds for reflection symmetry. The laws of electrodynamics are invariant under reflection as seen in the section 'Invariance of Laws Under Reflection' in this article above. This does not mean that the shapes of molecules governed by electromagnetic forces must necessarily be reflection symmetric. It is well-known that there exist the so-called chiral molecules where the mirror image of a molecule is not identical to the molecule, i.e, they cannot be superposed on each other, much like our left and right hands that are mirror images of each other. Chirality is an important feature of most biological molecules. Their existence does not necessarily imply violation of reflection symmetry in the basic laws that determine their structure and shape.

The discussion in this article is rather elementary limited to some examples of laws in mechanics and electrodynamics. However, reflection symmetry is a topic of great significance in modern fundamental physics. This is because reflection symmetry is true for all basic interactions of the elementary particles of nature, except the so-called 'weak interactions' (which govern β decay and other such processes). β-decay means emission of electrons (or their antiparticles called positrons) from unstable nuclei or in other processes involving elementary particles.

That this emission violates reflection symmetry was first seen in the β decay of polarised Cobalt-60 nucleus decaying to Nickel-60. A polarised nucleus is associated with a kind of angular momentum vector denoted by $\vec{J}$. The experiment studied the direction of emitted electron's momentum $\vec{p}$ relative to $\vec{J}$. The scalar product of the pseudo-vector $\vec{J}$ and the polar vector $\vec{p}$, $\vec{J} \cdot \vec{p}$ is a pseudo-scalar. If weak interactions responsible for β decay satisfied reflection symmetry, the average value of $\vec{J} \cdot \vec{p}$ should be zero. In the experiment, this was found NOT to be zero — more nuclei emitted electrons in the direction opposite to $\vec{J}$ than in the direction towards $\vec{J}$. The discovery of violation of reflection symmetry in weak interactions is among the great discoveries in physics of the last century.

Conclusion

This article gives only one illustration of how simple qualitative arguments in physics are useful. Such arguments do not involve much mathematics but are insightful and often aid in solving problems. They, of course, have limitations and cannot be a substitute for more detailed physics.

Several other kinds of simple arguments in physics include those based on dimensional analysis, scaling, symmetry and conservation laws, counting degrees of freedom in mechanics and thermal physics, intensive and extensive nature of quantities in thermodynamics, identifying dimensionless ratios for making approximations, going to limiting cases of general equations or theories as the first check of their validity, etc. Such arguments are found at numerous places in any good introductory physics text. Some of these were briefly outlined in the talk by the author referred earlier.