MULTILEVEL CONTOURS ON COMPLEX PLANES

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In this article, a new concept called multilevel contours on complex planes that was developed in 2021, is explained to senior high school students who are aware of the basics of real number systems and random variables. These ideas strengthen information geometry principles.

Keywords: Arcs, Contours, Length of an Arc, Continuous Functions

Introduction

Let $\mathbb{C}$ be the complex plane. Points in the complex plane are complex numbers which are defined as ordered pairs (*a,b*) of real numbers. Suppose z be a point in $\mathbb{C}$. The point z is denoted by,

$$z = (a,b) = a + ib \qquad (1)$$

In (1), $i = \sqrt{-1}$ and the real number a is thought of as a point on the $x - axis$ and the real number b is thought of as a point on the $y - axis$ (or also called in this case as imaginary axis). The number a is called the real part of z and the number b is called the imaginary part of z. If z_1 is a complex number such that $z_1 = (a_1, b_1)$, then $z = z_1$ if, and only if, $a = a_1$ and $b = b_1$. Complex numbers gave us several insightful analyses and practically useful applications. Studying complex planes in combination with other branches of mathematics has been fascinating.

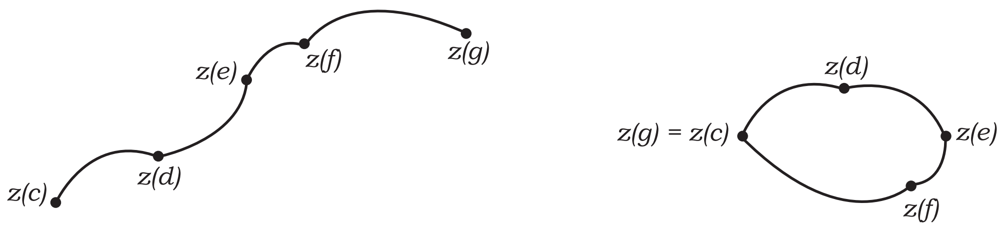
In 2021, a new concept called multilevel contours on complex planes was introduced by the author of this article and was published in 2022 with the title 'Multilevel Contours on Bundles of Complex Planes' in the *Geometry and Statistics*, Elsevier see Rao Arni SRS 2022. These news ideas were developed using geometry, analysis and probability models. Here, we learn about drawing such multilevel contours in an elementary approach.

Before we proceed to the new concept of multilevel contours, let us start with the basics of arcs and contours in complex planes.

An arc is a set of points $z = (a,b)$ in the complex plane such that:

$$z = z(t) = (a(t),b(t)) = a(t) + ib(t) \quad (c \le t \le d). \qquad (2)$$

Here c and d are real numbers, $a(t)$ and $b(t)$ are continuous functions of the real parameter t. Let us denote this arc by γ. If $z(t_1) \neq z(t_2)$ for two different parameters t_1 and t_2 on the real line, then we say γ is a simple arc. If $z(c) = z(d)$ we say γ is a Jordan curve. A contour in the complex plane is defined as an arc formed by joining piecewise smooth arcs joined end-to-end of piecewise arcs formed. See Figure 1.



(a) Contour

(b) Jordan Curve

Fig. 1: (a) A Contour is formed by joining four piecewise smooth arcs, $z = z(t)(c \le t \le d)$, $z = z(t)(d \le t \le e)$, $z = z(t)(e \le t \le f)$, c, d, e, f and g are real numbers. (b) A Jordan curve is formed by joining four piecewise smooth arcs, where $z(g) = z(c)$

Let L be the length of the contour that was formed by joining the piecewise arcs $z(c)$ to $z(g)$ in Figure 1 (a). The quantity L can be computed by,

$$L = \int_c^d L|z'(t)|dt + \int_d^e L|z'(t)|dt + \int_e^f L|z'(t)|dt + \int_f^g L|z'(t)|dt, \quad (3)$$

where $z'(t)$ is the derivative of $z'(t)$, and $|z'(t)| = \sqrt{|a'(t)|^2 + |b'(t)|^2}$ because $z'(t) = a'(t) + ib'(t)$. There are several textbooks available for further basics on complex numbers, arcs and contours. For example, students may refer to [Choudhary, 1992; Churchill, et al., 1984; Krantz, 2004; Pathak, 2019; Ponnusamy, 1995].

In the next few paragraphs, we will use the basics explained so far to form multilevel contours.

Let us consider five complex planes as shown in Figure 2. Suppose a complex number z_0 is chosen in $\mathbb{C}_1$ and a disk, say D_0 with a randomly chosen radius r_0 is drawn around z_0. A second random number within D_0 that is different from z_0 is randomly chosen and with another random radius, say r_1 a disk D_1 is formed with centre at z_1. An arc is constructed from z_0 to z_{01}. Similarly, a new arc is formed z_1 to z_2 by constructing a new disk D_2, and so on, to form new disks and new arcs.

When an arc, if it reaches the set $\mathbb{C}_1 \cap \mathbb{C}_5$, the next random number generated within a region $\mathbb{C}_1 \cap \mathbb{C}_5$ will have a scope to travel outside $\mathbb{C}_1$ and to reach $\mathbb{C}_2$. This travelling is possible due to creating the disks using the points in $\mathbb{C}_1 \cap \mathbb{C}_5$ and $\mathbb{C}_5$. Without having an intersecting plane $\mathbb{C}_1 \cap \mathbb{C}_5$, and not able to form arcs through the disks in $\mathbb{C}_1 \cap \mathbb{C}_5$, it is not possible to travel to $\mathbb{C}_2$ under the framework. Such piecewise arcs constructed could reach $\mathbb{C}_3$ (through $\mathbb{C}_2$) as well in a similar construction explained. We form a contour passing through more than two complex planes by joining the piecewise arcs end-to-end beginning from z_0 to z_1, z_1, and so on. A contour formed passing through multiple planes is called a multilevel contour. A more detailed description with a complete framework of the formation of multilevel contours with assumptions from probability principles can be seen in [Rao, 2022]. More technicalities could be beyond the scope of the present article for high school students. Such multivariate contours could be associated with passing information from one plane to another and extending the ideas of information geometry to complex plane geometry. The subject of information geometry was laid out by Rao, C.R. through his groundbreaking article in 1945 [Rao, 1945]. Read articles in references [Amari, 2021 and Plastino, 2020] for understanding the impact of [Rao, 1945] in an easily understandable manner. A recent article on the role of information geometry and complex planes in virtual tourism can be found in [Rao and Krantz, 2020].

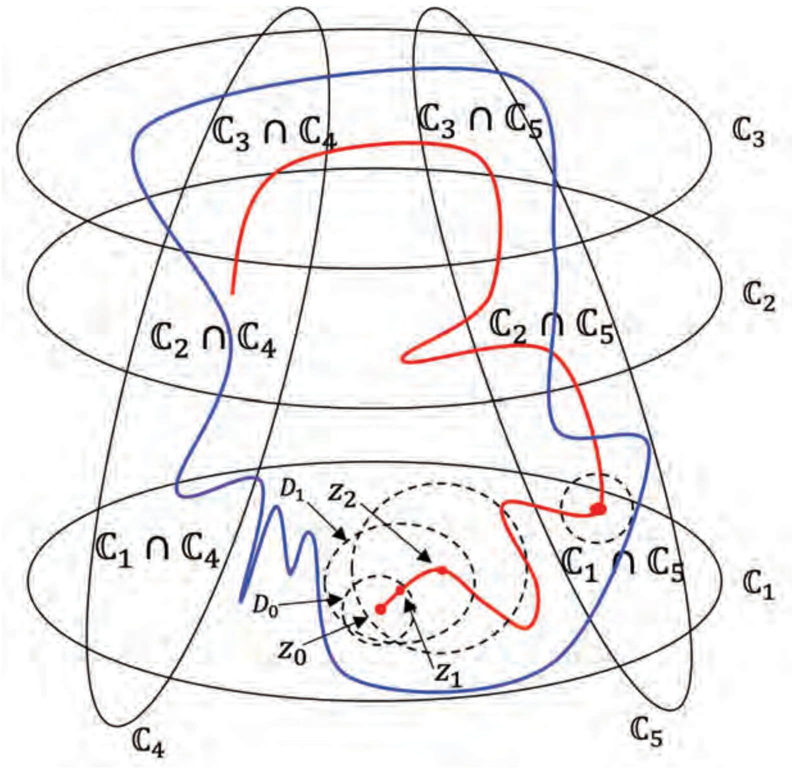


Fig. 2: Formation of multilevel contours passing through complex planes

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