MOTION OF A BALL FROM CREASE TO BOUNDARY IN A GAME OF CRICKET

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This is a model to calculate the motion of a ball hit by a batsman to score a boundary. The ball hits the ground, bounces several times and then grazes the ground to reach the boundary line evading intercept by any fieldsman. With given initial velocity, time taken by the ball to cross the boundary line covering the distance between the batsman and the ball touching the boundary line is determined. Herein the minimum initial velocity of the batted ball and its corresponding direction to strike a 'four' are also determined. Some numerical examples are also cited.

1. Introduction

The author earlier innovated two models of projectile motion of a cricket ball for 'bowled out' and 'caught out' respectively Maitra, 2007). In this design, a projectile motion of a cricket ball crashed by a batsman followed by its bouncing motion on the ground and thereafter rectilinear motion grazing the ground till it crosses or touches the boundary line, without being stopped by any fielder.

2. Equations of Motion of the Ball in Air

Let the batsman play a shot and the ball leaves the bat with a velocity u downwards at angle α to the horizontal.If it strikes the ground descending a height h and covering a horizontal distance R_0 in time T_0, escaping any intervention by a fieldsman, then its equations of motion in the air, whose resistance is neglected and where g is the acceleration due to gravitation, are given by

$$h = (u \sin \alpha)T_0 + \frac{1}{2} g T_0^2 \qquad (1)$$

$$R_0 = u \cos \alpha \, T_0 \qquad (2)$$

Eliminating T_0 betweeen (1) and (2) we can find

$$R_0 = (u/g)(-u \sin \alpha + \sqrt{u^2 \sin^2 \alpha + 2gh}) \cos \alpha \qquad (3)$$

$$T_0 = (-u \sin \alpha + \sqrt{u^2 \sin^2 \alpha + 2gh}) / g \qquad (4)$$

3. Equations of Motion of the Ball on the Ground

Overall motion of the ball discussed in this section consists of its successive bounces from the ground followed by its rectilinear motion on the ground till it reaches the boundary line to credit the batsman with a boundary ,i.e., four runs. Let the ball played by the batsman hit the ground with velocity v at angle β to the horizontal, the coefficient of elasticity between the ball and the ground being e; then one gets

$$v^2 \sin^2 \beta = u^2 \sin^2 \alpha + 2gh \qquad (5)$$

Because of gravity in the vertical direction which is also the line of impact, the horizontal component of the velocity of the ball is constant so that

$$u\cos\alpha = v\cos\beta \qquad (6)$$

Case 1. Theoretically the ball executes many rebounds, so to say infinitely many rebounds before its vertical component of the velocity vanishes. By Newton's law of collision, the vertical component of the velocity after first rebound from the ground is ev sinβ . Further in view of the velocity of rise being equal to the velocity of fall the subsequent vertical components of velocities of the ball due to successive rebounds, say, up to the nth rebound is given by

$$ev\sin\beta,\ e^2v\sin\beta,\ e^3v\sin\beta\ \ldots.\quad e^nv\sin\beta \qquad (7)$$

Since the time of fall is equal to the time of rise, the total time elapsed up to the nth rebound is given by

$$T' = 2(ev\sin\beta + e^2v\sin\beta + e^3v\sin\beta \ldots e^nv\sin\beta)/g$$

(vide G.P. series)

$$= \frac{2e}{g}.\frac{1-e^n}{1-e}v\sin\beta \qquad \text{(By use of ((5))}$$

$$= \frac{2e}{g}.\frac{1-e^n}{1-e}\sqrt{u^2\sin^2\alpha + 2gh} \qquad (8)$$

whereas the total distance described along the ground in this time is obtained as

$$R' = (u\cos\alpha)T' = \frac{2eu}{g}.\frac{1-e^n}{1-e}\sqrt{u^2\sin^2\alpha + 2gh}(\cos\alpha) \qquad (9)$$

If the horizontal distance from the just-ball of the boundary line compatible with motion of the smashed ball that reaches/crosses the boundary line after completion of n bounces be S_1 to score a boundary,

$$R_0 + R^1 \geq S_1 \qquad (10)$$

Case 2. While watching a cricket match, it is observed that the smitten ball moves on bouncing, stops bouncing and then moves on the ground in a rectilinear path to reach or cross the boundary line for 'four' runs.

Before the ball ceases to any further bounce, there arises a textbook problem of Dynamics, 3 giving infinitely many rebounds such that in consideration of (8) and (9) it covers a horizontal distance R1 in time T_1 due to bouncing:

$$R_1 = \lim_{n\to\infty} \frac{2eu}{g}.\frac{1-e^n}{1-e}\sqrt{u^2\sin^2\alpha + 2gh}(\cos\alpha)$$

$$= \frac{2eu}{g}.\frac{1}{1-e}\sqrt{u^2\sin^2\alpha + 2gh}\cos\alpha$$

$$(e<1, e^n \to 0 \text{ as } n\to\infty)$$

$$T' = \lim_{n\to\infty} \frac{2e}{g}.\frac{1-e^n}{1-e}\sqrt{u^2\sin^2\alpha + 2gh} \qquad (11)$$

$$= \frac{2e}{g}.\frac{1}{1-e}\sqrt{u^2\sin^2\alpha + 2gh}$$

After the ball stops bouncing, it begins to move on the ground obviously towards the boundary, of course, if any fields man is unable to stop it. Let f be the frictional resistance of the ground per unit mass of the ball that crosses the boundary line with velocity u0 travelling in this context a distance R_2 in time T_2 : (12)

$$R_2 = \frac{u^2\cos^2\alpha - u_0^2}{2f} \qquad (13)$$

$$T_2 = \frac{u\cos\alpha - u_0}{f} \qquad (14)$$

Thus in consonance with (3),(4),(11),(12),(13) and (14) the total horizontal distance travelled by the ball and the time reckoned from the instant of striking the ball by the batsman are given by

$$S = R_0 + R_1 + R_2 = [(-u\sin\alpha + \in \sqrt{u^2\sin^2\alpha + 2gh})/$$

$$g + \frac{u^2\cos^2\alpha - u_0^2}{2fu\cos\alpha}]u\cos\alpha \quad (15)$$

With $\in = \frac{1+e}{1-e}$

$$T = T_0 + T_1 + T_2 = (-u\sin\alpha + \in \sqrt{u^2\sin^2\alpha + 2gh})$$

$$g + \frac{u\cos\alpha - u_0}{f} \quad (16)$$

Or
$$s = (u\cos\alpha)\left(T + \frac{u_0}{f}\right) - \frac{u^2\cos^2\alpha + u_0^2}{2f} \quad (17)$$

Now let us express the velocity, i.e., exit velocity u of the ball from the contact with the bat in terms of the distance S:

$$\{\frac{g\left(s + \frac{u_0^2}{2f}\right)}{(u\cos\alpha)} + u\sin\alpha = \in \sqrt{u^2\sin^2\alpha + 2gh} + \frac{g}{2f}\cos\alpha$$

$$\left\{\frac{g\left(s + \frac{u_0^2}{2f}\right)}{(u\cos\alpha)} + u\left(\sin\alpha - \frac{g}{2f}\cos\alpha\right)\right\} = \in^2 \left(u^2\sin^2\alpha + 2gh\right)$$

which, because of substitutions

$$A = \frac{g\left(s + \frac{u_0^2}{2f}\right)}{(u\cos\alpha)}, B = \sin\alpha - \frac{g}{2f}\cos\alpha$$

$$C^2 = \in^2 \sin^2\alpha \text{ and } D = 2\in^2 ghl \quad (18)$$

turns out to be

$$\left(\frac{A}{U} + Bu\right)^2 = C^2u^2 + D$$

or, $(C^2-B^2)u^4-(2AB-D)u^2-A^2=0$, $C > B,\ 2AB > D, \in >$

$mu^4-2nu^2-A^2 = 0$

$$u = \sqrt{\frac{n + \sqrt{n^2 + m(A^2)}}{m}} \quad (19)$$

4. Maximum Distance Covered by the Ball During Successive Bounces

In this section is determined the maximum distance the ball can cover on the ground while it is bouncing 'up and down' and the optimum angle of striking to the horizontal by the batsman. Then with given initial velocity u, to crack a boundary, recalling (10) and (11), the following inequality holds

$R_0 + (R_1)max \geq S_1$ (20)

However for maximum or minimum of R_1

$$\frac{dR_1}{d\alpha} = 0 \quad (21)$$

or, equivalently $\frac{d(R_1^2)}{(\cos 2\alpha)} = 0$ (22)

And such (11) is rewritten as

$$\frac{R_1^2 g^2}{4e^2}(1-e)^2 = \frac{u^4}{4}\sin^2 2\alpha + ghu^2(1+\cos 2\alpha)$$

$$= \frac{u^4}{4}\left(1-\cos^2 2\alpha\right) + ghu^2(1+\cos 2\alpha)$$

so that by use of (22) one gets

$$\cos 2\alpha = \frac{2gh}{u^2} \qquad \cos\alpha_{opt} = \frac{1}{2}\cos^{-1}\left(\frac{2gh}{u^2}\right) \quad (24)$$

or, $\cos\alpha_{opt} = \sqrt{\frac{1}{2}\left(1 + 2\frac{gh}{u^2}\right)}$ $\sin\alpha_{opt} = \sqrt{\frac{1}{2}\left(1 + 2\frac{gh}{u^2}\right)}$

$$\frac{d^2R_1^2}{d(\cos 2\alpha)^2} < 0 \quad (25)$$

And hence by use of (11),(12) and (25), the maximum horizontal distance covered by the bounces is obtained as

$$(R_1)_{max} = \frac{e(u^2 + 2gh)}{g(1-e)} \quad (23)\ (26)$$

in time T_1 $$\alpha_{opt} = \sqrt{\frac{1}{2}\left(1 + 2\frac{gh}{u^2}\right)} \quad (26.1)$$

with fixed height and initial velocity u.

5. Minimum Velocity for a given Horizontal Distance to be covered by Bounces

We can show that with a fixed distance R_1 on the ground to be described by bounces of ball having its initial height h, there exist a minimum velocity of the ball and the corresponding angle of projection. So from equation (23),for maximum or minimum of u ie u^2 we get

$$\frac{d(u^2)}{(\cos 2\alpha)} = 0 \quad (27)$$

Hence differentiating (23) and using (27), one gets

$$ghu^2 - \frac{u^4}{4}(2\cos 2\alpha) = 0$$

$$u^2_{min} = \frac{2gh}{\cos 2\alpha_{opt}} \quad (28)$$

Now differentiating (23) twice with respect to (cos2α) subject to (27) we obtain

$$\left[\frac{u^2}{2}\left\{1-(\cos 2\alpha)^2\right\} + gh(1+\cos 2\alpha)\right]\frac{d^2(u^2)}{d(\cos 2\alpha)^2} = \frac{u^2}{2}$$

which implies $\frac{d^2(u^2)}{d(\cos 2\alpha)^2} \rangle 0$ (29)

which ratifies the minimum velocity given by (28)

To determine u_{min} and α_{opt} explicitly we employ (28) in (23):

$$\frac{R_1^2 g^2}{\alpha e^2}(1-e)^2 = u^2(1+\cos 2\alpha).$$

$$\left\{\frac{u^4}{4}(1-\cos 2\alpha) + gh\right\}$$

$$= u^2\left(1 + \frac{u^2}{2gh}\right)\left\{\frac{u^2}{4}\left(1 - \frac{u^2}{2gh}\right) + gh\right\}$$

$$= (u^2 + 2gh)^2 / 4$$

$$\frac{R_1 g(1-e)}{e} = u^2 + 2gh$$

$$u_{min} = \sqrt{\frac{g}{e}\{R_1(1-e) - 2he\}} \quad (30)$$

which in consequence of (28) gives

$$\cos^2\alpha_{opt} = \frac{2he}{R_1(1-e) - 2he} \quad (31)$$

$$\text{or, } \cos^2\alpha_{opt} = \frac{R_2(1-e)}{2\{R_1 g(1-e) - 2he\}}$$

$$\alpha_{opt} = \cos^{-1}\sqrt{\frac{(1-e)}{2\{R_1 g(1-e) - 2he\}}} \quad (32)$$

which suggests that if h=0 or h→0, $\propto_{opt} = 45^0$ or $\propto_{opt} \to 45^0$ then it reduces to a textbook problem. It is observed that with fixed velocity the maximum bouncing-horizontal- distance or with fixed bouncing-horizontal-distance the minimum velocity can be determined from either of equations (26) and (30).

6. Time Taken by the Ball to Reach the Boundary

Eliminating u_0 between (15) and (16) the time taken to reach the boundary line by the ball slapped by the batsman is given by

$$T = (-u\sin\alpha + \in \sqrt{u^2 \sin^2\alpha + 2gh})$$

$$\frac{1}{g} + \frac{1}{f}[u\cos\alpha - \{\frac{u^2}{2f}\cos^2\alpha - s - (\upsilon\sin\alpha + \in$$

$$\sqrt{u^2\sin^2\alpha + 2gh})\frac{u\cos\alpha}{g}\}^{\frac{1}{2}}\sqrt{2f}] \qquad e<1, \in>1 \qquad (33)$$

From, (33), it is ascertained that greater the horizontal component of the batted velocity, lesser the time for the ball to reach the bouncing.

7. Minimum Initial Velocity to Strike a 'Four'

At the sight of equation (15) or its another form, with given initial angle $\propto$, the velocity of the ball depends upon h,S, $\in$ and g. Nevertheless intuitively there exists angle $\propto_{opt}$ of projection by the batsman to achieve a 'Four' with a minimum-velocity of the ball, for which we need to put $\frac{du}{d\alpha}=0$ from (15) and then to find the corresponding value of $\propto$. But this gives a complicated equation involving sin $\propto$ and cos $\propto$. In order to avoid such complication we neglect h because h‹‹S in equation (15) which is rewritten as

$$\frac{A}{u^2} = (\in -1)\frac{\sin 2\alpha}{2} + \frac{\lambda}{2}(1+\cos 2\alpha) + \frac{\sqrt{2gh}}{2}(1+\cos 2\alpha)\in$$

$$\left(\text{writing } \sqrt{u^2\sin^2\alpha + 2gh} \cong u\sin\alpha + \sqrt{2gh\cos\alpha} \leq \sqrt{u^2\sin^2\alpha + 2gh}\right)$$

$$\text{or } \frac{2A}{u^2} = (\in -1)\sin 2\alpha + (1+\cos 2\alpha)\left(\lambda + \in \sqrt{2gh}/u\right) \qquad (35)$$

Where $A = g(S + \frac{u_0^2}{2f}), \lambda = \frac{g}{2f}$ (36)

For minimum or maximum of u, ie, for maximum or minimum of $\frac{1}{u^2}$ we have formed equation (35)

$$\frac{d\left(\frac{1}{u^2}\right)}{d\alpha} = 0$$

$$(\in -1)\cos 2\alpha - (\lambda + \in \frac{\sqrt{2gh}}{u_{min}})\sin 2\alpha \qquad (37)$$

$$\text{or, } \tan 2\alpha = \frac{(\in -1)}{\lambda + \in \frac{\sqrt{2gh}}{u_{min}}} = \mu \qquad (38)$$

Ultimately to evaluate u_{min} and $\propto_{opt}$ with desired accuracy a method of approximation is adopted.

Since h«S implying $\in \cdot \frac{\sqrt{2gh}}{u_{min}} << \lambda$, relation gives

$$\tan 2\alpha_{opt} = \frac{(\in -1)}{\lambda} \qquad (39)$$

so that $\sin 2\alpha = \frac{\mu}{\sqrt{1+\mu^2}}, \cos 2\alpha = \frac{\mu}{\sqrt{1+\mu^2}}$ (40)

which on substitution into (35) yields

$$u_{min}^2 = \frac{2\sqrt{1+\mu^2}A}{(\in -1)\mu + \left(\lambda + \in \frac{\sqrt{2gh}}{u_{min}}\right)(1+\sqrt{1+\mu^2})} \qquad (41)$$

Denoting

$(\in -1)\mu + \lambda\left(1+\sqrt{1+\mu^2}\right), \in\left(1+\sqrt{1+\mu^2}\right)$ and $2A\sqrt{1+\mu^2}$ by E,2F and G, respectively (41) reduces to a quadratic equation

$$Eu_{min}^2 + 2\sqrt{2gh}Fu_{min} - G = 0 \qquad (42)$$

whose solution gives

$$u_{min} = \frac{\sqrt{2gh}F + \sqrt{2ghF^2 + GE}}{E} \qquad (43)$$

which involves h, however small it is in comparison to S.

Substituting (41) into (38), we can obtain more accurate value of α_{opt}.

8. Numerical Examples

Rearranging (15) we can find the velocity u_0 with which the ball can reach the boundary line and using (16) the time taken to reach it.

$$u_0 = \left[\left(-u\sin\alpha + \in \sqrt{u^2 \sin^2\alpha + 2gh}\right) \frac{u\cos\alpha}{g} + \frac{u^2}{2f} \frac{\cos^2}{2}\alpha - S \right]^{\frac{1}{2}} \sqrt{2f} \quad (44)$$

Example 1. Let us suppose the initial velocity of the ball=u=25 metre per second, i.e., 90 kilometre per hour. Radius S of the cricket ground=65 metres. Then from (44) and (16) with some realistic values of $\in$ and α, the velocity with which the ball passes the boundary line=u0=22.47metre/sec in timeT=3.138 seconds.

But in case of the lifted batted- ball at angle α above the horizontal line,in the foregoing equations α is to be replaced by--α.

Example 2. With u=20 metre/sec, i.e., 72 kms/hour, similarly,u_0=11.97metre/sec, T=4.57seconds.

Example 3. With u=30 metre/second,i.e., 72 km/hours, similarly u_0=27.47,T=2.58 seconds.

Example 4. With u=35 metre/sec , i.e., 126km/hour, similarly u_0=2 metre/ses T=4.16 seconds.

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