# Problem-based Learning in Basic Physics – X

**A.K. Mody**
C-14 Beverly Hills, Lam Road
Devlali, Nashik 422401
Maharashtra

**H. C. Pradhan**
HBCSE, TIFR, V. N.
Purav Marg, Mankhurd
Mumbai – 400 088

In this article (10th in the PBL series) we consider various examples from different areas of physics where we have systems in which one of the parameter varies continuously and to estimate any effect we need to take differential element and perform integration separately (different from that which is already performed in arriving at standard textbook formula).

**Key Words:** [In this article we present problems for a problem-based learning course based on system with continuously varying parameter.]

1. A block of mass *m* sliding on a frictionless horizontal surface experiences an air resistance proportional to its velocity. If it has initial speed $v_0$, find an expression for its velocity as a function of time.

**Solution**

Acceleration $a = \frac{dv}{dt} = -\alpha v$

$$P = F.v = m\frac{dv}{dt}v$$

Integrating, $v = v_0 \exp(-\alpha t)$

2. A car starts from rest and start moving while its engine continues to deliver constant power *P*. Find the distance travelled by the car in time *t*.

**Solution**

In this case, since constant power is delivered, acceleration is not constant.

Power delivered $P = F.v = m\frac{dv}{dt}v$

$\therefore mvdv = Pdt$ ; integrating which, we get

$$\frac{v^2}{2} = \frac{p}{m}t$$

$\therefore \frac{dx}{dt} = \sqrt{\frac{2Pt}{m}}$ integrating which, we get

$$x = \left(\frac{2}{3}\sqrt{\frac{2P}{m}}\right)t^{3/2}$$

3. A cylindrical solid of mass $10^{-2}$ kg and cross-sectional area $10^{-4}$ m$^2$ is moving parallel to its axis (the *x*-axis) with a uniform speed of $10^3$ m/s in the positive direction. At $t=0$, its front face passes the plane $x=0$. The region to the right of this plane is filled with dust particle of uniform density $10^{-3}$ kg/m$^3$. When a dust particle collides with the face of the cylinder, it sticks to its surface. Assuming that the dimension of the cylinder practically remains unchanged and the dust sticks only to the front face of the *x*-coordinate of the front of the cylinder. Find the *x*-coordinate of the front of the cylinder at $t=150$ s.

[JEE 1993]

**Solution**

In this case we have an object (cylinder) whose mass keeps on increasing (variable mass problem) as $m = m_0 + \rho Ax$ and equation will be $m\frac{dv}{dt} = -v\frac{dm}{dt}$ If we assume mass of the cylinder as $m_0$ and velocity at $t$ =0 as $v_0$, then integration yields $v = \frac{m_0 v_0}{(m_0 + \rho Ax)}$. The same would also follow from conservation of total momentum.

Further integration yields,

$$\frac{1}{2}\rho Ax^2 + m_0 x - m_0 v_0 t = 0$$

Upon substitution of parameters gives $x=10^5$ m at $t$ = 150 s.

4. A uniform chain of length $L$ is kept on a smooth frictionless horizontal table with small length hanging from the edge. This makes chain slide down from the table. Find the speed when it just loses contact with the table.

**Solution**

Initially, let a small length hang from the table. Thus gravity pulling it down with the acceleration $a = g\frac{x}{L}$

$$\therefore \frac{dv}{dt} = g\frac{x}{L} \quad \therefore \frac{dv}{dx}\frac{dx}{dt} = g\frac{x}{L}$$

$$\therefore \frac{x}{L}vdv = \frac{g}{L}xdx$$

Integrating we get $v^2 = \frac{g}{L}x^2$

When chain just leaves the table, its vertical length $x \to L \therefore v = \sqrt{gL}$ is the speed when it just loses contact with the table.

5. Calculate the gravitational potential energy of a uniform spherical distribution of mass M and radius R.

**Solution**

When the a mass $dm$ comes from infinity to the surface of sphere of mass $m$ and radius $r$, its potential energy is $dU = -\frac{gmdm}{r}$

When mass accumulates and grows from zero to $M$ and radius from zero to $R$, potential energy becomes $U = \int_0^M -\frac{Gmdm}{r}$

Since mass distribution is uniform, density

$$\rho = \frac{M}{\frac{4\pi}{3}R^3} = \frac{m}{\frac{4\pi}{3}r^3}$$

This gives $dm = \rho 4\pi r^2 dr$ and integral becomes $U = -\frac{16\pi^2}{3}G\rho^2\int_0^R r^4 dr$

Which upon integration gives

$$U = -\frac{16\pi^2}{15}G\rho^2R^5 = -\frac{3}{5}\frac{GM^2}{R}$$

[Calculation of electrostatic potential energy of a charged sphere is identical to the one discussed here for gravitational potential energy.]

6. For a particle performing SHM with time period $T$, calculate time required to travel from $a/2$ to $a$, where $a$ is the amplitude.

**Solution**

For a particle performing SHM,

velocity $v = \omega\sqrt{a^2 - x^2}$ where *x* is displacement and $\omega = \frac{2\pi}{T}$,'

Thus, $\frac{dx}{dt} = \omega\sqrt{a^2 - x^2}$

which gives $\int \frac{dx}{\omega\sqrt{a^2 - x^2}} = \int dt$

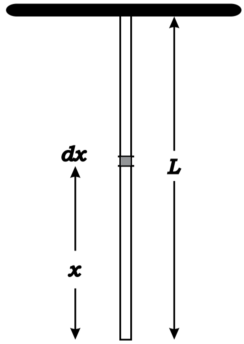
$$\therefore \Delta t = \frac{1}{\omega}\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_{a/2}^{a} = \frac{1}{\omega}\left[\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{\omega}\left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \frac{\pi}{3\omega}$$

$$\therefore \Delta t = \frac{T}{6}$$

7. A uniform rope of mass 0.4 kg and length 2.45 m hangs from a ceiling.
   (i) Find an expression for increase in length of the rope.
   (ii) Find the speed of the transverse wave in the rope at a point 0.5 m distance from the lower end.
   (iii) Calculate the time taken by a transverse wave to travel the full length of the rope. Derive the formula used. [Take $g$ = 9.8 m/s$^2$, and $Y = 2\times10^{11}$N/m$^2$]. [(ii) and (iii) Roorkee1991]

**Solution**




Consider an element of length *dx* at a distance *x* from the lower end of the rope.

The stress experienced by this element is given by *pxg* where *p* is the density of the rope.

If length of this element increases by $d\Delta x$, then strain$= \frac{d\,\Delta x}{dx} = \frac{\text{stress}}{Y} = \frac{\rho g x}{Y}$. Here *Y* is the Young's modulus of the material of the rope.

∴ increase in length of this element

$$d\,\Delta x = \frac{\rho g x}{Y}dx$$

∴ increase in length of the rope

$$\Delta x = \int_0^L \frac{\rho g x}{Y}dx = \frac{\rho g L^2}{2Y} = \frac{MgL}{2AY}$$

where *M* is the mass of the rope and *A* is its area of cross-section.

(ii) velocity of the wave at a distance *x* from the lower end $v = \sqrt{\frac{T}{m}} = \sqrt{\frac{\rho g x A}{\rho A}} = \sqrt{gx}$

here $T$ is the tension at a distance $x$ from the lower end and $m$ is linear density of the rope.

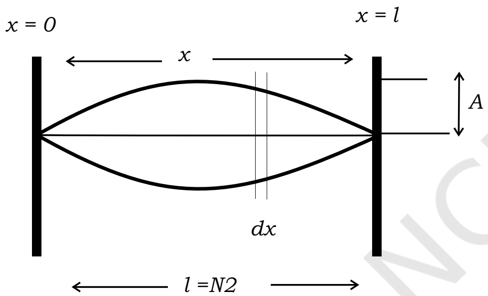
(iii) time taken by wave to travel the rope

$$\Delta t = \int_0^L \frac{dx}{v} = \int_0^L \frac{dx}{\sqrt{gx}} = 2\sqrt{\frac{L}{g}} = 1\text{sec}$$

8. A string tied between $x = 0$ and $x = l$ vibrates in fundamental mode. The amplitude $A$, tension $T$ and mass per unit length $\mu$ is given. Find the total energy of the string.

[JEE 2003]

**Solution**




The amplitude at any point $x$ is given by $a$=$A$ sin

$kx$ where $k = \frac{2\pi}{\lambda} = \frac{\pi}{l}$

Where $\omega = 2\pi n = \frac{\pi}{l}\sqrt{\frac{T}{\mu}}$ , mass of string element of length $dx$ is $dm=\mu dx$

Energy of this element while oscillating is

$$dE = \frac{1}{2} dm\,\omega^2 a^2$$

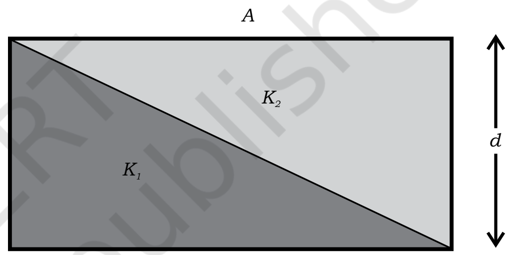
Thus total energy of the string

$$E = \int_0^l dE = \int_0^l \frac{1}{2}\mu \frac{\pi^2 T}{l^2 \mu} A^2 \sin^2\left(\frac{\pi}{l}x\right)dx$$

Which upon integration yields $E = \frac{1}{4}\pi^2 A^2 \frac{T}{l}$

9. The capacitance of a parallel plate capacitor with plate is $A$ and separation $d$ is $C$. The space between two wedges of dielectric constants $K_1$ and $K_2$, respectively. Find the capacitance of the resulting capacitor.

[JEE 1996]




**Solution**

Let $A$=$bL$, where $b$ is the width of the capacitor into the plane of the paper.

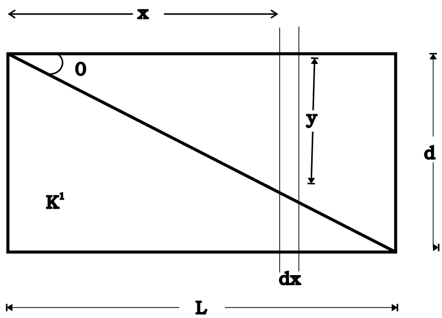
Consider a strip of length $dx$ at a distance $x$ from one end as shown below.

The corresponding thicknesses are $y$ and

$(d–y)$. Notice here that $\tan\theta = \frac{d}{L} = \frac{y}{x} = \frac{d-y}{L-x}$

Each of these strips have capacitance

$$C_1 = \frac{K_1\varepsilon_0 bdx}{(d-y)} \quad \text{and} \quad C_2 = \frac{K_1\varepsilon_0 bdx}{y}$$




The series combination gives capacitance of the entire strip $dC$ given by

$$\frac{1}{dC} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d(L-x)}{K_1\varepsilon_0 bLdx} + \frac{xd}{K_2\varepsilon_0 bLdx}$$

$$= \frac{d}{\varepsilon_0 A}\left[\frac{L-x}{K_1} + \frac{x}{K_2}\right]\frac{1}{dx}$$

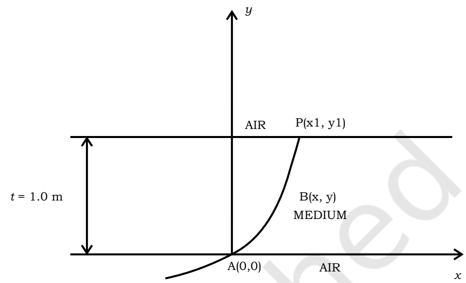
$$= \frac{d}{\varepsilon_0 A}\left[\frac{(K_1 - K_2)x + +K_2 L}{K_1 K_2}\right]$$

The capacitance $C$ of the entire capacitance therefore can be obtained by integrating $dC$.

$$\therefore C = \int_0^L dC = \frac{K_1 K_2 \varepsilon_0 A}{d}\int_0^L \left[\frac{1}{(K_1 - K_2)x + K_2 L}\right] dx$$

$$= \frac{K_1 K_2 \varepsilon_0 A}{d(K_1 - K_2)} \ln\left(\frac{K_1}{K_2}\right)$$

10. A ray of light travelling in air is incident at grazing angle (incident angle ≅ 90°) on a long rectangular slab of a transparent medium of thickness $t = 1.0$ m (see the figure). The point of incidence is the origin $A(0, 0)$. The medium has a variable index of refraction $\mu(y)$ given by $\mu(y) = [ky^{3/2} + 1]^{1/2}$, where $k = 1.0$ (metre)$^{-3/2}$.




The refractive index of air is 1.0

(a) Obtain the relation between the slope of the trajectory of the ray at a point *B (x,y)* in the medium and the incident angle at that point.

(b) Obtain the equation of the trajectory *y (x)* of the ray in the medium.

(c) Determine the coordinates *(x,y)* of the point P, where the ray intersects the upper surface of the slab boundary.

(d) Indicate the path of the ray subsequently.

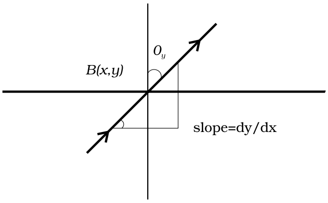
[JEE 1995]

**Solution**

Note here that refractive index $\mu$ depends on co-ordinate $y$ only.

(A) For a series of interfaces, Snell's law is $\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2 = \ldots = \mu_n \sin\theta_n$ .

Thus at the point of incidence and point B, the relation is $\mu_1 \sin\theta_1 = \mu_y \sin\theta_y$ and $\theta_1 \approx 90°$.




(B) Which (referring to figure gives $\mu(y)\sin\theta_y$ = 1 and slope = $dy/dx$ = tan (90 -$\theta_y$) = cot$\theta_y$

$$\therefore \frac{dy}{dx} = \cot\theta_y = \sqrt{\mu^2 - 1} = \left[ky^{3/2}\right]^{1/2}$$

which upon integration yields $y = k^2\left(\frac{x}{4}\right)^4$.

(C) Point P is at $y_1$ = 1.0 m which gives $x_1$ = 4.0 m

(D) At point A and P $\mu_A \sin\theta_A = \mu_P \sin\theta_P$ where $\theta_A \approx 90°$where as both $\theta_A = \mu_P = 1$ as subsequently at P ray enters air. This gives $\theta_P \approx 90°$

## References

JEE: *Joint Entrance Examination for admission to* IIT.

IIT ROORKE: *Roorke University Entrance Examination (Now IIT*-Roorke)