PROBLEM-BASED LEARNING IN BASIC PHYSICS – IX

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In this article, (ninth in the series), we present problems for a problem-based learning course based on the order of magnitude, approximations and errors.

Introduction

In this article, ninth in the Problem-based Learning (PbL) series, we consider various examples from different areas of physics where we often make approximations. We are trying to discuss the meanings of these approximations.

Approximations with Functions

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \ldots \approx 1 - x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \ldots \approx 1 + x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \approx x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \approx 1 - \frac{x^2}{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + - \ldots \approx 1 + x$$

For all the above approximations to be valid, we must have $x << 1$.

The limit of approximation is defined by the value of the first term which is ignored with reference to the actual value. For example, in $e^x \approx 1 + x$, if the term $x^2/2$ is smaller than the accuracy we are looking for, then the term and higher order terms can be ignored.

Consider the following examples from physics to understand the importance of the above mentioned approximations and the order of magnitude calculations.

1. **We say $\sin\theta = \theta$ for small angle where θ is in radians. Discuss up to what value of θ is this approximation valid? Check for same value of θ, if the approximation is valid for $\cos\theta \approx 1 - \theta^2/2$.**

   Discussion

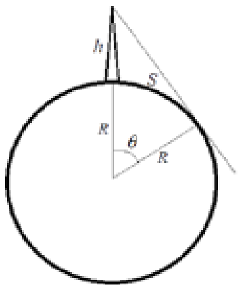
   Let us consider

   $$\theta = 30^0 = \pi/6 \ radian = 0.523598$$

   for which $\sin\theta = 0.5$. Thus, for at 30°, difference between $\sin\theta$ and θ is about 4%. Thus, depending on the accuracy we are looking for, we should say $\sin\theta$ and θ lie within that many per cent. In this case, the second term in the expansion becomes, $\theta^3/6 = 0.024$ which now, at 30°, $\cos\theta = 0.866025$ and $1 - \theta^2/2 = 0.862922$. Here, the error would be about 0.35%. Next term, $\theta^4/4 = 0.0031$. Check similar difference for 10°.

2. **Assuming earth to be a perfect sphere of radius 6,400 km and Mt. Everest to have a height of 8 km, find the greatest distance on earth's surface from where Mt. Everest will be visible.**

Discussion



Consider the diagram shown in the figure below.

Here, *h* is the height of the mountain and R the radius of the earth. If a tangent is drawn from the peak of the mountain to the earth's surface, then S represents the maximum distance on the earth's surface from where the mountain would be visible [assuming (that a symbol) the earth's surface to be a perfect sphere].

Referring to the figure : $R=(R+h)\cos\theta$ Considering $h << R$ we can get $\cos\theta = 1 - h/R$ Comparing this with approximation $\cos\theta \gg 1-\theta^2/2$, we get $\theta^2 = 2h/R = 16/6400 = 1/400$ i.e. $\theta = 1/20$ radian or $9/\pi$ degree. This gives $S = R\theta = 6400/20 = 320\text{km}$ Note that here we used approximation $\frac{1}{1+x} \approx 1-x$, where $x=h/R$ Here, $x^2 = (1/800)^2$ and higher order terms can justifiably by neglected.

3. **Calculation of g: When we calculate g near the earth surface, we take $g = \frac{GM}{R^2}$ where $M = 6\times10^{24}\,\text{kg}$ and $R = 6.4\times10^6$ m. Why do we ignore the effect of the sun and moon near the earth surface, despite the fact that the sun is huge?**

Discussion

Let us check the contribution of each near the earth's surface.

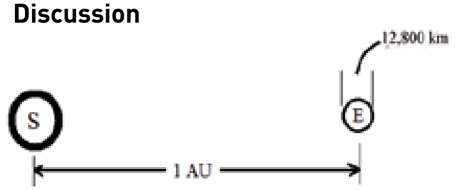
For earth $\frac{M}{R^2} = 1.46\times10^{11}$ SI units

For sun near earth $\frac{M_{sun}}{r^2} = 8.89\times10^7$ SI units

For moon near earth $\frac{M_{moon}}{r^2} = 4.97\times10^5$ SI units , using the data given in Example 4 below. Thus, contribution of the sun is 4 orders and that of moon is 6 orders smaller than that due to earth. Hence, for practical purposes, we ignore the effect due to the sun and the moon. However, when it comes to force on large masses as the earth and its ocean, even a tiny effect can prove to make a significant difference and due to the size of the earth can cause tides in the ocean (see Example 4 below).

4. **What causes tides? We want to understand tides in the ocean based on Newton's law of gravitation. Light takes 500 seconds to travel from the sun to the earth and the diameter of the earth is 6,400 km. From this information, find out the difference in gravitational pull (acceleration) experienced by a body when it is near the sun to that when it is far. [Given: $g = 6.67\times10^{11}\,\text{Nm}^2/\text{kg}^2$ and mass of the sun = 2×10^{30} kg]**

Discussion



$$g_{near} = \frac{GM}{(r-R)^2} = 0.005928889\text{m}/s^2$$ and

$$g_{far} = \frac{GM}{(r+R)^2} = 0.0059278772\text{m}/\text{s}^2$$

Which gives $\Delta g = 1.011\times10^{-6}\,cm/s^2$.

Considering the mass of the oceans, which is a fraction of the earth, this generates a huge difference in the force causing tides.

Now, from the mass of the moon = 7.3483×10^{22} kg and earth-moon distance = 3,84,400 km calculate similar differential g, (say Δg) at earth due to moon. Considering the relative position of the earth, sun and moon, can you estimate the magnitude that causes tidal effect on the earth on a new moon day, full moon day and first or third quarter? Here, assume that sun-earth-moon are along the same line on a new moon and full moon day (not in the same order as you can understand). What can happen if they align on the same line? Take the radius of the sun to be 7×108 m and that of a black hole to be $\mathrm{R} = \frac{GM}{c^2}$, where c is the speed of light. Take the average height of a person to be 1.5 m. Calculate differential g between the head and the toe of an average height person on the surface of the earth, sun and black hole of one solar mass. This tidal gravity on the earth acts on the ocean which is a huge mass (that of ocean), say 100th of that of the earth. Can you calculate this tidal (differential) force on this mass on earth? Estimate this tidal (differential) force that a normal human (average mass 50 kg) would experience (difference of g between head and toe) on the surface of (i) sun (ii) 1 solar mass black hole. [Use a calculator and preserve at least 8 digits after decimal for all your calculations].

5. **Why water vapour at 100°C burns the skin whereas at room temperature it does not?**

Discussion: The average kinetic energy of a gas molecule is considered to be of the order of kBT where $k_B = 1.38 \times 10^{-23}$. J/K is the Boltzmann constant and T is the absolute temperature. Accordingly, water molecule (of water vapour) at room temperature (300 K) has kinetic energy $k_B T \approx 1/40\mathrm{eV}$, whereas at $400\mathrm{K}, k_B T \approx 1/30\mathrm{eV}$. Can you explain why at 300 K water vapour is harmless, whereas at 400 K it feels hot and can even burn the skin?

[Hint: the distribution of molecules ΔN in the energy range between E and E + ΔE is given by Maxwell-Boltzmann distribution function: $\Delta N = \frac{2\pi}{(\pi k_B T)^{3/2}} N e^{-E/k_B T} E^{1/2} \Delta E.$ For simplification, you may assume energy at $E \approx 1\ \mathrm{eV}$ (energy just sufficient to cause burn) in the interval $\Delta E \approx k_B T.$ [You should get about 10^4 times more molecules having energy 1 eV at 400 K compared to that at 300 K].

6. **Estimation of the size of an atom**

Discussion: This means mass of one aluminum atom is $\mathrm{m}_{Al} = \frac{M_A}{N_A}$, where, NA = 6.02×1023/mole is the Avogadro number. This means, the volume occupied by one atom (assuming cubic arrangement) is $V = a^3 = \frac{m_A}{p} = 16.6 \times 10^{-30}\ m^3$. This gives $a = 2.554 \times 10^{-10}\,\mathrm{m} = 2.554\ \mathrm{A^o}$. Here 'a' can be treated as diameter of the atom. Actual crystal structure of aluminum is FCC that gives 4 atoms per cubic cell which modifies the calculation to get a = 4.05 Ao. The separation of atoms in FCC gives the diameter of the atom. Thus, aluminum atom radius is estimated to be 1.43 Ao. We can safely consider typical atomic radius to be of the order of 1 Ao.

7. What can exist inside the nucleus?

Discussion: Consider an aluminum atom which has an atomic weight A = 27. The radius of a nucleus is given by $R = R_0 A^{1/3}$. Where R0 = 1.22 Fermi. [1 Fermi = 10 -15 m]. This gives $R_{Al} \approx 4$ fm. Any particle — proton, neutron or electron, if inside the nucleus will have uncertainty in position of the order of 4 fm. From uncertainty principle, this gives, $p = \Delta p \approx \frac{h}{\cdot} = 1.66 \times 10^{-19}\, kg - m/s$. For electron, this gives speed greater than that of light. Which means electron will have energy $E = pc = 5 \times 10^{-11}\,\text{J} \approx 47\,\text{MeV}$, whereas for proton and neutron

$$E = \frac{p^2}{2m} \approx 1\,MeV$$

Electrons observed in decay have much less energy , and thus cannot exist inside the nucleus as independent particles. Proton and neutron, on the other hand, can exist inside the nucleus.

8. Discussion of Simple Harmonic Motion (SHM)

In case of a pendulum, the differential equation is solved for small amplitude taking approximation sin $\theta \approx \theta$. Thus, expression for period obtained would be approximate within those appropriated limits.

In case of spring, we estimate,

$$T = 2\pi\sqrt{\frac{m}{k}}$$

assuming spring to behave as elastic for extension and compressions produced. For a perfectly elastic spring potential energy,

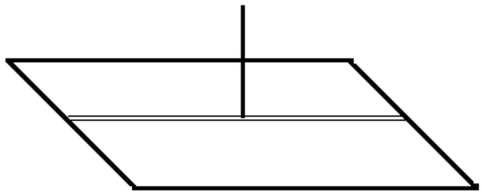
$$V = \frac{1}{2}kx^2$$

whereas in reality, spring can neither be compressed by indefinite amount nor extended. Thus, **V** must have higher order terms and should have expression of the type $V = \frac{1}{2}kx^2 + ax^3 + bx^4 + \ldots$

Here, the values of 'a' and 'b' are small compared to 'k' and thus, the corresponding terms become important only when 'x' is large. Time period is also accurate within this approximation.

9. Infinity in Physics: Examples from Electrostatics and Magnetostatics

Discussion: In electrostatics, we deal with infinite line of charge, infinite charge plane, and in magnetism, we deal with infinite straight wire carrying current. In real life, we do not have such infinite objects. Then, why do we derive such expressions? In each of the above mentioned case, we find an expression for field at some distance from these infinite objects. Consider an infinite plane. For any point outside this plane, we can have two elements exactly at the same distance from the foot of the perpendicular,



on the plane contributing to the field. The field contributions from these two elements have two components each. The components parallel to the plane cancel each other and the one perpendicular to the plane add up and survive. Thus, field will always be perpendicular to the plane.

What if we have a finite size plate? Even in this case, if we are sufficiently close to the plane surface (distance is very small compared to the dimensions of the plate), the result of infinite plane applies. In case of parallel plate capacitor, if the separation between plates is too small compared to the size of the plate, field lines are all perpendicular except near edges. These effects would not be negligible if plate separation is comparable to the size of the plates.

We encourage readers to construct similar line of arguments for linear charge distribution and straight wire carrying current.

10. Infinity in Exponential Equations

Now, consider famous equations like

(i) $N=N_0e^{-\lambda t}$ from radioactive decay

(ii) $q = Q\left(1-e^{(-t/RC)}\right)$ for charging of a capacitor in an RC circuit

(iii) $q = Q\left(1-e^{(-t/RC)}\right)$ for growth of current in an LR circuit

Here $\frac{1}{\lambda}$, RC, and $\frac{L}{R}$ have dimensions of time and are called time constants. In each case, we can write time constant as τ and exponential term can be represented as $e^{-t/\tau}$.

We say that as $t \to \infty, e^{-t/\tau} \to 0$.

What does infinite time mean here?

Consider the values

$$e^{-1} = 0.3679,\ \ e^{-2} = 0.1353,$$
$$e^{-3} = 0.0498,\ \ e^{-4} = 0.0183,$$
$$e^{-5} = 0.0067.$$

Thus, $t \to 5\tau$ is good enough for infinity at 0.7% accuracy and system would stabilise for all practical purposes as the measuring instrument may not be sensitive enough to record 0.7% variation. Otherwise, we need to consider higher values of *t*.

Note: Even when we say $e^x \to 1$ as $x \to 0$, $\left(\text{since } e^x \approx 1+x\right)$, the value of x determines the limit of this approximation.

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