UNIFORM CIRCULAR MOTION

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Introduction

The concept of displacement as a vector quantity and the methods of adding and subtracting vectors are used to study motion of objects in two and three dimensions. Normally motion in two dimensions is studied first and then extended to three dimensions. motion using general vector notations without separating it into two and three dimensions. It would therefore be valid for both the cases.

We begin with the concept of average velocity (an extension of the one-dimensional case, where one has to consider as many components as there are dimensions):

$$\vec{v}_{av} = (\vec{r}_2 - \vec{r}_1)/(t_2 - t_1) \equiv \Delta\vec{r}/\Delta t \qquad (1)$$

Where $\Delta\vec{r}$ or $(\frac{\Delta y}{\Delta x})$ is the displacement (from position $\vec{r}_1$ to $\vec{r}_2$) that occurs in a time interval Δt or $(t_2 - t_1)$.

In two and three dimensions, the average speed over any time interval will usually be greater than the magnitude of the average velocity over the same time interval since the actual path of the object between the end points of the displacement is curved. For example in a circular path (1/2 complete revolution) distance covered is $2\pi r$ while displacement covered is 2r as it is clear from Fig. 1. Thus distance→displacement and therefore speed → velocity.

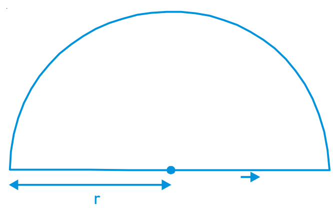

Fig.1

Comparison or order relation exists for similar quantities only. For scalars there is no order relation but for vectors order relation is there, e.g. $\vec{A} > \vec{B}$ or $|\vec{A}| \rightarrow |\vec{B}|$.

> How do average speed and average magnitude of velocity compare in one dimension? The students and even teachers have a tendency to take them to be equal. Let them consider various kinds of one-dimensional motion and discover that the equality is not always valid.

Instantaneous velocity is defined as usual using the limiting procedure:

$$V = \lim_{\Delta t \to 0} (\Delta\vec{r}/\Delta t) = d\vec{r}/dt \,^{o}\, v(t). \qquad (2)$$

This has no direct experimental significance in kinematics. If an object is traveling along a curve in a plane or in a three-dimensional space, the instantaneous velocity at any point on the path has the direction of the tangent line at that point.

Acceleration is again an extension of the corresponding expression in one dimension

$$\vec{a} = \lim_{\Delta t \to 0}(\Delta \vec{v} / \Delta t) = d\vec{v} / dt \qquad (3)$$

Analogous to equations for one dimension, kinematics equations for two and three-dimensional motion with constant acceleration $\vec{a}$ are

$$\vec{v} = \vec{v}_0 + \vec{a}t, \qquad (4)$$

$$v^2 = v_0^2 + 2\vec{a}.(\vec{r} - \vec{r}_0), \qquad (5)$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \vec{a}t^2 / 2. \qquad (6)$$

Common misconceptions

- Students often think that $\Delta\vec{r}$ is in the same direction as $\vec{r}$ and $\Delta\vec{v}$ is in the same direction as $\vec{v}$, most likely an incorrect extension of the idea in one dimension. On further extrapolation, these would mean that $\vec{v}$ is in the same direction as $\vec{r}$ and $\vec{a}$ is in the same direction as $\vec{v}$ which are not true in general in two and three dimensions.

  This can be rectified with the help of Fig. 2 in which $\vec{r}$ and $\Delta\vec{r}$ have been shown in a general case. They have clearly different directions.

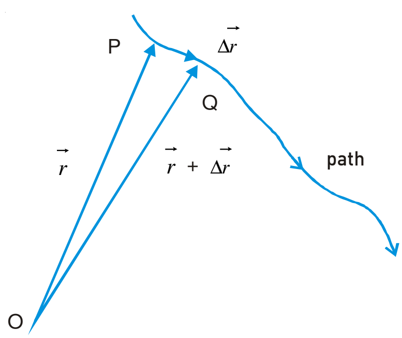

Fig.2

Here P and Q are the positions (described by the vectors $\vec{r}$ and $\vec{r} + \Delta\vec{r}$) of the body at instant t and $t + \Delta t$ respectively along the path indicated. By the triangle law of vector addition

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$

With reference to this diagram and using their knowledge of vector algebra, the students should be able to show that if $\vec{v}$ and $\vec{v} + \Delta\vec{v}$ be the velocities of the body at P and Q respectively. $\Delta\vec{v}$ and hence $\vec{a}$ will have a direction different from that of $\vec{v}$.

From this they should conclude that in general, a vector $\vec{A}$ and its time derivative $d\vec{A} / dt$ need not point in the same direction. The students should also recognize that the line representing position $\vec{r} + \Delta\vec{r}$ is not necessarily longer than that representing position $\vec{r}$ in spite of the positive sign in the former expression.

Another interesting feature they may note is that in order to describe a motion in two dimensions, four initial conditions (initial position and initial velocity) are needed. The number increases to six in three dimensions, and reduces to two in one dimension. Let the students discover these numbers using their knowledge of vectors and their components.

Uniform Circular Motion

At higher secondary level, uniform circular motion is usually studied as an example of a periodic or repetitive motion in a plane, neglecting the gravitational attraction of the earth. This simple ideal motion introduces important concepts like those of *centripetal* and *centrifugal* forces. It is observed that students as well as teachers have a number of conceptual problems and alternative conceptions in this case. We had glimpses of some of these in our interaction with students and teachers. Some studies have also discovered other hard spots and alternative conceptions.

Hard Spots

Common hard spots for students include linear velocities (tangential/radial), axis of rotation, the concept of angular speed vis-à-vis angular frequency and vectorial nature of angular displacement and angular velocity.

Alternative Conceptions

- Uniform circular motion means *constant velocity* along a circle.
- A particle or a body executing uniform circular motion is *in equilibrium.*
- Tension (in the string to which the rotating body is tied) *balances the* centripetal force.
- Centrifugal force *cancels* or *balances* the centripetal force.
- Centrifugal forces are *real i.e. the effects are real.*
- An object moving in a circle with constant speed has *no or zero* acceleration.
- An object moving in a circle will continue in *circular motion* when released.
- An object in circular motion will fly out *radially* when released.

Remedial Suggestions

Here we would like to discuss some aspects of uniform circular motion keeping in mind the hard spots and alternative conceptions noted above.

We consider a particle traveling round a circular path of raidus *r* with a constant speed v. This constitutes an example of uniform circular motion. It is important for the students to note that it is not a case of constant velocity or constant acceleration. The particle takes a time *T* (measured in s), called the period of the motion, to go once round the circle. Obviously the line joining the position of the particle to the centre of the circular path sweeps out the angle of 2π radians (rad) in time interval *T and so an angle* of $2\pi/T$ *radians per second or* rad s^{-1}. This is defined as the angular speed or angular frequency ω of the particle. Though the unit of angular speed is rad/s, the second name of ω (angular frequency) is also appropriate, considering the fact that *rad* is a dimensionless unit. Both the names of ω are used in literature. In fact, some call it angular velocity too without caring for the distinction between scalar and vector. Clearly ω as defined above is a scalar constant and is given by

$$\omega = 2\pi / T = 2\pi /(2\pi r / v) = v / r$$

Here v *is also the magnitude of instantaneous velocity.* The students ought to recognise that these are the characteristics of a uniform circular motion and they need not worry about vectors like position vector, linear velocity, angular velocity, etc. to begin with. First, they should be able to form a mental picture of these quantities *r, n* and *w* and show them on a planar diagram depicting the uniform circular motion. This will help them understand the respective vectors $\vec{r}, \vec{v}$ *and* $\vec{\omega}$ as and when they appear.

We shall now introduce these quantities as vectors using the concepts that have already been developed while dealing with two-dimensional motion in this section. The approach would not only consolidate the students' concept of vectors but would also bring out the vectorial nature of the various quantities clearly. This may be used in conjunction with the treatment available in standard textbooks.

We shall refer to Fig. 3 for uniform circular motion where both the Cartesian *(x, y)* and plane polar coordinates (r, θ) have been shown in (a) and their respective unit vectors in (b). *O* is the observer of the motion stationary at the centre of the circle.

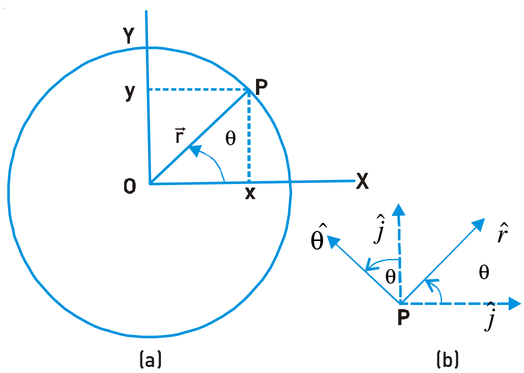

Fig. 3

Here it is to be noticed that

1. A uniform circular motion is an accelerated motion.
2. When centripetal force is zero, a body cannot move on a circular path.

> It is not necessary to show everything in one diagram as it is likely to make the diagram look clumsy and inconvenient. For better clarity and convenience, students should be encouraged to draw as many diagrams as required.

With reference to Fig.3 (b) we have

$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$, and (7)

$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta.$ (8)

since $\vec{r} = r\hat{r}$, the velocity vector becomes

$$\vec{v} = \frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} = r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = r\omega\frac{d\hat{r}}{d\theta} \quad (9)$$

$$= r\omega\hat{\theta} = v\hat{\theta}$$

Where, we have used the fact that $\omega = d\theta / dt$ and $\frac{d\hat{r}}{d\theta} = \hat{\theta}$ [from equations (7) and (8)]. Clearly, equation (2c) implies that velocity in this case is purely tangential (denoted by $\hat{\theta}$) in nature and has no radial component (why?). Next, the acceleration vector is given by

$$\hat{a}_{cp} = \frac{d\vec{v}}{dt} = v\frac{d\hat{\theta}}{dt} = v\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = v\omega\frac{d\hat{\theta}}{d\theta} \quad (10)$$

$$-v\omega\hat{r} = -a\hat{r}$$

In which $d\hat{\theta}/d\theta = -\hat{r}$ [from equations (2a) and (2b)]. Clearly, equation (2d) implies that acceleration is radial and points towards the centre. This is the centripetal acceleration whose magnitude is $a = v\omega = v^2 / r = \omega^2 r$. If m be the mass of the particle, then the force acting on the particle is the centripetal force $\vec{F}_{cp}$ that is given by the formula

$$\vec{F}_{cp} = m\vec{a}_{cp} = -ma\hat{r} = -(mv^2 / r)\hat{r} = -(m\omega^2 r)\hat{r} \quad (11)$$

It is important to recognise that this is the force that must act on the particle in order to keep it moving uniformly in the circular path.

> The observer *O* therefore concludes that $\vec{F}_{cp}$ is the net force that acts on the particle, and hence, the particle is not in equilibrium by definition. It thus follows that uniform circular motion cannot occur without force, unlike uniform rectilinear motion for which no force is needed (first law of motion).

The point is, there must be some source or agent to supply the force $\vec{F}_{cp}$. The agent may be the tension in the string tied to a rotating stone or the gravitational attraction of the sun in case of a planetary body, e.g. a vehicle moves uniformly on a circular level road, it is the frictional force between the tyres of the vehicles and the road that provides the necessary centripetal force.

The agent thus *supplies or provides* the centripetal force necessary for the circular motion. It *does not cancel* the centripetal force as is often thought. If it were to cancel the centripetal force the net force would be zero and particle will move in a straight line, which will contradict the conclusion stated in the previous box.

Let us now look at the motion of the particle from a different perspective. Let the observer *O'* rotate together with the particle at the same uniform rate *w*. Clearly, the particle is at rest and in equilibrium relative to this observer; this means that a force equal in magnitude to the centripetal force and opposite in direction must be acting on the particle along with the centripetal force. The new force

$$\vec{F}_{cf} = -\vec{F}_{cp} = -m\vec{a}_{cp} = ma\hat{r} = (mv^2/r)\hat{r} = (m\omega^2 r)\hat{r} \quad (12)$$

is termed the centrifugal force. Let us consider a situation in which two passengers are sitting side by side in a bus. One of them is sitting rather 'tightly' holding on to the frame of the seat, whereas the other passenger is sitting 'loosely'. Then the bus takes a sharp turn to the left. The first passenger will observe that his companion is shifting on the seat away from him towards the right. This is an effect of the centrifugal force, which we might have experienced on several occasions without realising it. The first passenger's head might also tilt to the right because by rigid structures fixed to the seat frame, then the effect of the centrifugal force cannot be 'observed' unlike with human beings, but it will still exist in the frame of reference of the bus when it takes a turn.

From this discussion it becomes clear that centrifugal force has interesting and unique characteristics:

(i) Centrifugal force and centripetal force do not constitute an action-reaction pair because they act or supposed to act on the same body as observed from a rotating frame (observer *O'*) or a non-inertial frame of reference. In such a frame, these two forces may be said to cancel each other as we saw above.

(ii) Centrifugal force exists in a non-inertial frame of reference and has no existence in an inertial frame (observer *O*).

(iii) Centrifugal force does not arise from any interaction unlike the centripetal force, which needs to be provided from interaction with an agent such as the string.

(iv) For reasons such as above, centrifugal force is often termed a 'pseudo-force' or 'fictitious force'. However, its effect can be *real* and *seen* by an observer in a non-inertial frame of reference.Recognizing these would help rectify misconceptions regarding centrifugal force.

The interesting question of 'what happens when a body in circular motion is released' is answered by noting that for an observer like *O* (in an inertial frame of reference), the body becomes free as no force is acting on it now, and as is well known.

Newton's first law of motion takes over. The students should be encouraged to work out the rest leading to the answer. It would be interesting and instructive for the teacher to first work out the case when the observer would continue to rotate (like the observer *O'*) after the body is released, and then instruct the students accordingly.

In our discussion so far we have not used the vectorial nature of quantities like angular displacement and angular velocity and the concept of axis of rotation, which are not adequately understood by the students. But they have their own importance. Angular velocity is denoted by the vector $\vec{\omega}$ whose magnitude *w* is called angular speed or angular frequency. By convention, direction of $\vec{\omega}$ is defined by the *right hand rule*. Let the right hand be curled such that the fingers point in the direction of rotation of the particle (clockwise or anticlockwise). $\vec{\omega}$ will then be taken to point in the direction of the extended thumb. The direction of the extended thumb is also taken to represent the axis of rotation. The same conclusion may also be reached by using the familiar *right hand screw rule.* Let the students verify this.

If we write $\vec{\omega} = \omega\hat{\omega}$, then $\hat{\omega}$ will lie along the axis of rotation and will be given by the right hand rule. Since $\omega = d\theta / dt$ and angular velocity is defined as the rate of change of angular displacement (analogous to the case of rectilinear notion), we can define the angular displacement vector $\vec{\theta} = \theta\hat{\omega}$. It is important to note that it is $\hat{\omega}$ and not $\hat{\theta}$ [$\hat{\theta}$ gives the direction of the tangential velocity $\vec{v}$ as in equation (9) that gives the direction of $\vec{\theta}$. The students should verify that $\vec{\theta}$ is actually a vector in two dimensions since (i) it has both magnitude and direction, and (ii) it obeys the commutative law of vector addition.

The students ought to recognise that (i) above is just a necessary but not sufficient condition for a quantity to be termed a vector. It has to satisfy condition (ii) also.

This should not, however, generalize the vectorial nature of $\vec{\theta}$ from two to three dimensions since the commutative law of vector addition is not valid for finite angular displacements in three dimensions. (Let the students actually verify this.) Nevertheless infinitesimal angular displacements (like $d\theta$) satisfy the requirement of commutativity and therefore qualify as vectors in three dimensions.

We have now three vectors associated with uniform circular motion: $\vec{r}, \vec{v}$ and $\vec{\omega}$. Let the students do the exercise of finding the vectorial relationship among them using the well known relation $v = r\omega$.

We may thus note that uniform circular motion or rotational motion is unlike the case of uniform rectilinear motion and is replete with many interesting and intriguing features. This encourages us to compose a *six-line limerick* as follows:

Mr. Particle' takes a ride,
Glee and surprises he can hardly hide.
He goes straight and yet turns left,
Moves uniformly and still accelerate.
Loose in seat, he's pushed to right;
Cut loose, becomes Newton's delight

The limerick may be used as a teaching learning aid by the teacher and students alike as hidden in it are important concepts of rotation.

Conclusion

In this section, we have discussed some important aspects of motion in two and three dimensions with the objective of helping teachers develop the correct concepts in the learners. Uniform circular motion has been discussed in some detail as an example of two-dimensional motion. Learners' hard spots and alternative conceptions have been pointed out. This may be fruitfully used along with conventional textual material. The students ought to realise that mental imagery of three-dimensional space as well as the skill of representing a three-dimensional motion on the plane of paper are required for developing the right concepts. The teacher can definitely do her/his bit for this.

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