THE MYSTERIOUS INFINITY

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The concept of endlessness of numbers arises very early when a child in fact learns about the numbers like 1, 2, 3 ... etc., called natural number. A precocious child can argue whether there is any largest natural number. He can build up his argument by assuming that he has found the biggest number N. However, just by adding 1 to N will fetch him another number, N+1, which is still bigger. In this way, he will finally end up with the concept of numerousness of endlessness of numbers.

This endlessness is what the mathematicians have named infinity. It isn't a number like 1, 2 or 3. In fact, it is hard to say what is exactly. It is even harder to imagine what would happen if one tried to manipulate it using the arithmetic operations that work on numbers. For example, what if one multiplies it by 2? Is 1 plus infinity greater than, less than, or the same size as infinity plus 1? What happens if one subtracts 1 from it? These and many other related questions centre round the weird concept of infinity, which has a mysterious realm of its own.

The usual symbol for infinity is ∞. This symbol was first used in a seventeenth century treatise on conic sections. It caught on quickly and was soon used to symbolize infinity or eternity in a variety of contexts.

The appropriateness of the symbol ∞ for infinity lies in the fact that one can travel endlessly around such a curve called leminiscate (Fig.1). Endlessness is, after all, a principal component of one's concept of infinity. Other notions associated with infinity are indefiniteness and inconceivability.

Fig. 1

In the physical world there are various sorts of infinities that could originally exist e.g., infinite time, infinitely large space, infinite-dimensional space, infinitely continuous space etc. Georg Cantor, a German mathematician, has dubbed such infinites physical infinities. In addition to these infinities, Cantor also defines two other types of infinities. These are absolute infinite and mathematical infinities.

Cantor, in fact, allowed for many intermediate levels between the finite and the absolute infinite. These intermediate stages correspond to what Cantor called *transfinite* numbers, that is, numbers that are finite but nonetheless conceivable.

The word 'Absolute' in the context or absolute infinite is used in the sense of 'non-relative, nonsubjective'. An absolute exists by itself and is the highest possible degree of completeness (in religion, this absolute corresponds to God!).

Cantor, introduced the concept of sets and showed that only real numbers can be expressed in terms of infinite sets. A set, as defined by mathematicians, is any collection of distinct or well-defined objects of any sort—people, pencils, numbers. Different objects such as a book, a football, a glass and a table can also constitute a set. The objects constituting a set are called its elements or members. An infinite set is one whose number of elements cannot be counted.

Let us define N as the finite set of all natural numbers:

N = [1, 2, 3, 4,...]

If we remove from N the infinite set of odd natural numbers, then we will be left with the infinite set of even natural numbers:

E = [2, 4, 6, 8, ...]

Similarly, taking away from N the infinite set of even natural numbers leaves us with the set of odd natural numbers:

O = (1, 3, 5, 7...)

So, we see that when an infinite subset is removed from an infinite set the result is an infinite set. Now, consider the infinite set of all natural numbers starting from, say, 11 onwards and remove it from the infinite set of all natural numbers. You will be left with finite set of first ten natural numbers. Thus, removing an infinite subset from an infinite set can also result in a finite subset.

The set of natural numbers and the set of odd (or even) natural numbers are both finite. But, which of the two represents bigger infinity? If we attempt to count the number of odd natural numbers by the usual way, then we find that a one-to-one correspondence exists between the natural numbers and the odd natural numbers. This means that corresponding to every natural number n there exists an odd natural number 2n-1:

1 2 3 4 5,..... n .....

↕↕↕↕↕ ↕

1 3 5 7 9,..... 2n - 1.....

Thus, the number of odd natural numbers is infinite and equal to the number of natural numbers. This is expresses by saying that the cardinality of the set of odd natural numbers is the same as the set of all natural numbers. These two sets, therefore, represent the same infinity.

It can be easily seen that the set of perfect squares has also the same cardinality as the set of natural numbers:

1 2 3 4 5 ..... n, ...

↕↕↕↕ ↕ ↕

1 4 9 16 25,..... n^2, ...

Similarly, the sets of cubes, fourth powers, fifth powers etc., of natural numbers can be seen to have the same cardinality as the set of natural numbers. All these sets, therefore, represent the same infinity.

It can be shown that even the set of all rational numbers has the same cardinality as the set of natural numbers. We can list all the rational numbers by first displaying then in an array. Along the first row we list all those which have numerator 1, along the second row all those with numerator 2, along the third row all those with numerator 3, and so on (Fig.2).

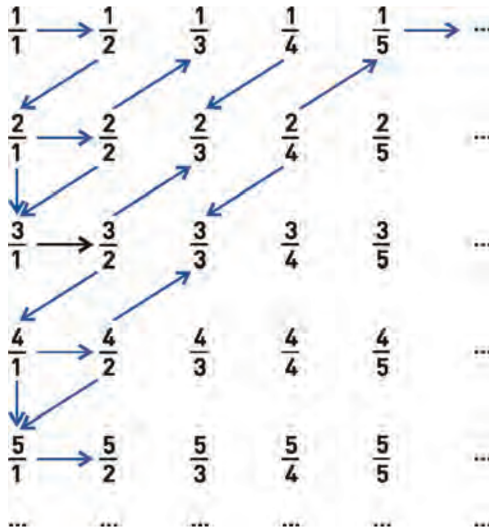
$$\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\[4pt]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\[4pt]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} \\[4pt]
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} \cdots\cdots \\[4pt]
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5}
\end{array}$$

Fig. 2

Now, if we decide to count row by row then we will be exhausting all the natural numbers by one-to-one correspondence with the entries in the first row. It might seem therefore that because there are an infinity of rows, there must be more rational numbers than natural numbers.

However, we can arrange to count in a different way, beginning with $\frac{1}{1}$ continuing with the rational numbers whose numerator and denominator add upto 3, then with those whose numerator and denominator add upto 4, and so on. This gives us a diagonal method of counting (Fig.3).

It is clear that this arrangement of diagonal counting enables us to list al the natural numbers. The fact that some of them occur more than once in different forms such as $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots$ and $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$, does not affect the force of the argument.



In this manner, all the rational numbers can be counted i.e., they can be put into one-to, one correspondence with the natural numbers. There are therefore the same number of rational numbers as there are natural numbers. So, the set of rational numbers has the same cardinality as the set of natural numbers.

We may denote the infinity of natural numbers by N_0 (Cantor used the Hebre letter $\aleph_0$ (aleph-null) to denote this infinity). We say that the set of natural numbers has the same cardinal N_0.

We have seen that the sets of all odd and all even natural numbers have the same cardinality (or infinity) as the set of all natural numbers. Thus, the sets of odd and even natural numbers have N_0 members each. As the sets of odd and even natural numbers together go to make up the set of natural numbers we have: $N_0 + N_0 = N_0$

This defies the dictum applicable to finite collections that 'a part cannot be equal to the whole'. A strange result: isn't it? But, an infinite collection (set) is a collection of objects equal in number to only a part of itself.

Let us now consider a real number i.e., a number expressible in the form:

$\pm n.r_1r_2r_3r_4$

where n is any natural number and r's are digits from 0 to 9. This decimal may be finite and terminating, or may be non-terminating but recurring or may be non-terminating and non-recurring. Cantor expressed real numbers in terms of infinite sets and showed that the real numbers cannot be put in one-to-one correspondence with the natural numbers. Thus, the infinity or cardinality of real numbers is not the same as that of natural numbers.

If we define the infinity of real numbers by N_1 then we can say that N_1 is greater than N_0 provided that we take care in defining precisely what we mean by 'greater than' in the context of infinite numbers. We do this by noting that whereas N_0 numbers can be put in one-to-one correspondence with any part of the N_1 real numbers, there is no way in which N_1 numbers can be put in one-to-one correspondence with a part of N_0 numbers. This applies, of course, to any collection of N_0 and N_1 objects. It is a fundamental truth of what we call 'set theory', the general theory which derives from Cantor's work, and is not confined to collection of numbers.

Numbers such as N_0 and N_1 are called 'transfinite' i.e., they are beyond the finite yet there are infinitely many of them. This was proved by Cantor by showing that given any number, finite or transfinite, it is always possible to construct one that is greater.

Thus, as proved by Cantor, an infinite number of transfinite numbers viz. N_0, N_1, N_2...etc., can exist. We, therefore, have an infinity of infinities. But, can there also exist an infinity between N_0 and N_1, between N_1 and N_2 between N_2 and N_3, and so on. According to Cantor's *continuum hypothesis* there is no infinity in between. Generalising the hypothesis we can say that no infinity can exist between N_k and N_{k+1}.

Cantor's work has now received proper recognition. However, initially, it was put to server criticism. Some of his contemporaries described his set theory as 'a disease', 'repugnant to common sense', and so on. These attacks depressed Cantor and led to a series of nervous breakdowns. This genius dies in 1918 in a mental institution in Halle. The following remarks about infinity made by Cantor deserves special mention:

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds".