SRINIVASA RAMANUJAN : THE MATHEMATICIAN

J.N. Kapur

Honorary Visiting Professor and Senior Scientist, INSA

Introduction

It is an undisputed fact that Srinivasa Ramanujan (1887-1920) was the greatest Indian mathematician of modern times. He was also one of the greatest mathematicians of all times in the world.

For assessing the greatness of Ramanujan, we have to answer the following questions:

- What is the field of investigation of mathematicians? What are the typical problems they discuss?
- How does a mathematician think and work?
- What are the qualities of first-rate mathematician?
- What are the qualities of a great mathematician?

We shall attempt to answer these questions with the help of Ramanujan's work on partitions and try to do this at a level at which any intelligent person with a knowledge of high school mathematics can appreciate the answers.

A mathematician studies *patterns* in number and geometrical forms. The number he studies may be natural numbers, integers, rational numbers, real numbers, complex numbers, ordered pairs of thee numbers, ordered n-triples of these numbers and so on...... The geometrical forms may be curves or surfaces in two or three or n-dimensional spaces and generalisations of these.

A mathematician first works out special cases and then with his experience and intuition, he looks for patterns. He then makes conjectures and verifies them with more special cases. When he is convinced about a pattern, he tries to prove it precisely, logically and rigorously.

Thus a mathematician has to have a strong power of intuition and also a strong power of deductive reasoning.

An applied mathematician observes similar patterns in nature and society and he also needs strong power of intuition and logical deduction.

Some mathematicians have strong intuitive powers and some have strong deductive reasoning powers and some have both.

Both intuition and logical reasoning are natural gifts, but these can be developed by training and concentration.

We can rate every mathematician in the scale of 1 to 100 for intuition and on a similar scale for deductive powers.

Professor Hardy, who in some sense discovered the genius of Ramanujan for the world, tried to rate some mathematician in the first scale for intuition and natural talent for mathematics. He gave himself a rating of 25, his friend and colleague Littlewood a rating of 30, David Hilbert a rating of 80 and Ramanujan a rating of 100.

Hardy, Littlewood and Hilbert may however be rated higher than Ramanujan in the scale for deductive powers. The reason for this can be partly in the long training they had received and partly in the Western traditions of rigorous deductive reasoning. Ramanujan had also begun developing his deductive powers and had he lived a little longer, he might have surpassed them in deductive reasoning powers also. However that was not to be.

There is no doubt that in his genius, his natural talent, his unbelievable intuition for mathematical results and his great powers of concentration, he may be ranked as almost the greatest mathematician of all times.

To illustrate how the mind of a mathematician works and in particular how Ramanujan's mind worked, we consider some examples from theory of partitions.

A Systematic Formula for the Number of Partitions of a Natural Number

A partition of a natural number n is a sequence of non-decreasing positive integers whose sum is n. The total number of partitions of n is denoted by p(n). Thus we have,

one method of finding p(n) is the 'brute force' method. One simply enumerates all partitions. One may not be able to proceed beyond 10 or 20 by this method. One will require a great power of concentration and even then one is likely to miss some partitions. Even if a person has perfect powers of concentration and writes one partition

n	Partitions											P(n)
1	1											1
2	2	11										2
3	3	21	111									3
4	4	31	22	211	1111							5
5	5	41	32	311	221	2111	11111					7
6	6	51	42	411	33	321	3111	222	2211	21111	111111	11

The number p(n) increases fast with n. Thus we have the Table I:

n	10	20	30	40	50	60	70	80
p(n)	42	627	564	37338	204206	966467	4087968	15796474
n	90	100			200	600	1000	5000
p(n)	56634173		190569292		3972999029388	$0(10^{21})$	$0(10^{31})$	$0(10^{75})$

per second, one will take about 126,000 years to write all partitions of 200; and to write all partitions of 5000, the whole age of Universe will not enough!

However mathematicians observe patterns in partitions, develop formulae and can write all partitions of n in a much shorter time. Thus Prof. Hans Raj Gupta prepared tables of partitions upto n = 600 without the use of computers. This was a great achievement in itself.

Ramanujan asked a basic question

"Can we find p(n), without enumerating all the partitions of n?" and he and Hardy gave the first answer,

$$p(n) \simeq \frac{1}{4n\sqrt{3}} e\pi(2n/3)^{\frac{1}{2}} \qquad (1)$$

where π is the ratio of the circumference of a circle to its diameter and has an approximate value of 22/7. The number e is a constant whose approximate value is 2.71828. Also the symbol $\simeq$ stands for 'asymptotically equal to'. This does not give the exact value of p(n), but it gives values which are better and better approximations to p(n) as n increases. In fact, we get the Table II:

It is seen that the difference between the exact and asymptotic values goes on increasing, though the percentage error goes on decreasing. It can be shown that the percentage error becomes smaller and smaller as n becomes larger and larger. In fact, the percentage error can be made as small as we like by making n sufficiently large.

It was a remarkable achievement of Ramanujan to think of this asymptotic formula for p(n). He thought of this remarkable result without any formal training in mathematics beyond the 'Intermediate stage' (equivalent to senior secondary) and without any help from anyone. He learnt mathematics himself; he formulated the problem himself and then in collaboration with Hardy, he obtained the formula for giving the correct order of magnitude of p(n) for large values of n.

Most mathematicians work in the same way in which Ramanujan did. They have to learn the mathematics needed for research themselves, they have to identify significant problems for themselves, they have to make conjectures and then they have to prove them. However they do this with the help of libraries well furnished with books and journals, with the help of research

n	10	20	30	40	50	60
asymptotic value given by (1)	48	492	6080	40081	2175967	1024034
% error	14.26	10.37	8.49	7.35	6.55	5.95
n		70	80	90	100	200
asymptotic value given by (1)	4312796	16607269	5936950	199286739	4.1003717 × 105	
% error	5.50	5.13	4.87	4.57	3.36	

guides, with the help of discussions with peers, with the help of conferences and symposia and so on. What is remarkable about Ramanujan is that he did all this by himself. He had no teacher, no research guide, no library worth the name, no journals, no discussions, no seminars except at a late stage, yet he obtained wonderful results.

Ramanujan-Hardy Formula for the Exact Value of Number of Partitions of a Natural Number

Ramanujan was not satisfied with the asymptotic expression given in (1) (which we may denote by $p_0(n)$ and was convinced that there must be an exact expression for p(n) of the form

$p_0(n) + p_1(n) + p_2(n) + p_3(n) + \ldots\ldots\ldots\ldots$

where $p_0(n) \gg p_1(n) \gg p_2(n) \gg p_3(n)$ ..... (Here $\gg$) stands for much greater than) so that the terms decrease fast and first few terms may give the correct value for p(n). In fact he and Hardy collaborated together to find this series. For n = 200 their series gives

$p_0(200) = 3972998993185.896$

$p_1(200)$ =	+ 36282.978
$p_2(200)$ =	- 87.555
$p_3(200)$ =	+ 5.147
$p_4(200)$ =	+ 1.424
$P_5(200)$ =	+ 0.071
	3972999029387.961

Thus six terms are sufficient to give us the exact value. For n = 1000, one may require 15 terms or so, but they showed that the number of terms required will be always of the order $\sqrt{n}$.

The Hardy-Ramanujan theorem for p(n) is one of the most remarkable theorems in mathematics. At this stage we cannot do better than quote Littlewood (Maths Gazette 14, 427-428, 1929).

"The reader does not need to be told that this is a very astonishing theorem and he will readily believe that the method by which it was established involved a new and important principle, which has been found very useful in other fields. The story of the theorem is a romantic one.......One of Ramanujan's Indian conjectures was that the first term of the formula was a very good approximation to p(n). The next step in development, not a great one, was to find the solution as an asymptotic sum of which a fixed number of terms were to be taken, the error being of the order of the next term. But from now to the very end, Ramanujan insisted that much more was true than had been established. "There must be a formula with error O(1)". This was his most important contribution; it was both absolutely essential and most extraordinary. The number of terms was made a function of n; this was a very great step and involved new and deep function-theory methods that Ramanujan obviously could not have discovered by himself. The complete theorem then emerged. But the solution of the final difficulty was impossible without one more contribution from Ramanujan, this time a perfectly characteristic one.......His suggestion of the right form of function to be used was an extraordinary stroke of formal genius without which the complete result can never come into the picture at all. There is indeed a touch of real mystery. Why was Ramanujan so sure about correct functional form needed?

Theoretical insight to be the explanation, had to be of an order hardly to be credited.....There is no escape from the conclusion that the discovery of the correct form was a single stroke of insight. We owe the theorem to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic and most fortunate work that was in him. Ramanujan's genius did have this one opportunity worthy of it".

We have quoted Littlewood extensively for the following reasons:

(i) It shows the extraordinary genius, intuitive powers and great mathematical insight of Ramanujan.

(ii) It shows the interaction between intuition and deductive reasoning in mathematics and shows that each need the other. The image of the mathematics as a purely deductive science is incomplete, if not completely wrong.

(iii) It shows the remarkable phenomenon that to prove a result concerning integers, function theory methods based on the concepts of complex numbers of the form $x + \sqrt{-1}\, y$ were required. Thus, imaginary numbers are essential for proving 'real' results.

(iv) Mathematics is a great intellectual enterprise and the proof of every great result in mathematics involves a great intellectual effort and great intellectual struggle. Most of the proof are published without giving an insight into the struggles which go into obtaining these. Here is an exceptional example.

(v) Hardy-Ramanujan collaboration led not only to the formula for p(n), but also to the development of 'circle method' needed in proving these formulae. The circle method has been fundamental to many other problems of analytical number theory where methods of analysis are used to prove results in number theory.

Ramanujan's Contributions to Congruence Properties of Partition Function

By looking carefully at the tabulated values of p(n) from n = 1 to 200, Ramanujan noted the following patterns:

(i) $p(5n + 4)$ is always divisible by 5 so that p(4), p(9), p(14), p(19),... are divisible by 5.

(ii) $p(7n + 5)$ is always divisible by 7 i.e., p(5), p(12), p(19), p(26),... are divisible by 7.

(iii) $p(11n + 6)$ is always divisible by 11 i.e., p(6), p(17), p(28), p(39),... are divisible by 11.

(iv) $p(25n + 24)$ is always divisible by 25 i.e., p(24), p(49), p(74), p(99),... are divisible by 25.

(v) $p(35n + 19)$ is always divisible by 35 i.e., p(19), p(54), p(89), p(124),... are divisible by 35 and so on .

We are giving in appendix the values of p(n) from n = 1 to 100 and readers will find it interesting to verify the properties given above.

Ramanujan proceeded to prove the first four of these properties and then conjectured the following theorem:

"$p(5^a 7^b 11^c\, n + \lambda)$ is divisible by $5^a 7^b 11^c$ for n = 0, 1, 2, 3...

if $24\lambda - 1$ is divisible by $5^a 7^b 11^c$"

As particular cases of this result, we have

(i) a = 1, b = 0, c = 0 gives $p(5n + \lambda)$ is divisible by 5 if $24\lambda - 1$ is divisible by 5 i.e., if $\lambda = 4$.

(ii) a = 0, b = 1, c = 0 gives $p(7n + \lambda)$ is divisible by 7 if $24\lambda - 1$ is divisible by 7 i.e., if $\lambda = 5$.

(iii) a = 1, b = 0, c = 0 gives $p(25n + \lambda)$ is divisible by 25 if $24\lambda - 1$ is divisible by 25 i.e., if $\lambda = 24$.

(iv) a = 0, b = 2, c = 0 gives $p(49n + \lambda)$ is divisible by 49 if $24\lambda - 1$ is divisible by 49 i.e., if $\lambda = 47$.

The reader will find it interesting to verify it for all the values of n = 1 to 100. In fact Ramanujan found that it was true for all n = 1 to 200. This does not however prove that the conjecture is true. In fact Ramanujan was himself careful to state that "The theorem is supported by all the available evidence, but I have not yet been able to find a general proof".

His attitude was that of a mathematician for whom no amount of empirical evidence is enough. There are results which are correct upto 10^9 or 10^{12} and fail thereafter.

In the present case, the conjecture was proved to be false only after 12 years when in 1930, Chawla noticed from the tables prepared by Hans Raj Gupta that –

p(243) = 1339782599344888 is not divisible by 343.

Now if we put a = 0, b = 3, c = 0 in Ramanujan's conjecture, we have first to find λ so that $\lambda - 1$ is divisible by 343 and we find 24 X 243 – 1 = 5831 = 17 X 343 is divisible by 343 so that $\lambda = 243$. If we put n = 0 in Ramanujan's conjecture then p(243) should be divisible by 343 but it is not (you may verify it).

That Ramanujan's conjecture was proved false is no reflection on Ramanujan's genius. In fact such conjectures provide a great incentive for mathematical research.

Ramanujan's conjecture was no exception. Forty-eight years after Ramanujan published his original conjecture, Atkin proved the modified conjecture, viz.,

"$p(5^a7^b11^c + \lambda)$ is divisible by $5^a7^{(b+2/2)}11^c$ if $24\lambda - 1$ is divisible by $5^a7^b11^c$".

Thus Ramanujan's conjecture is valid for all positive integral values of a and c, but is valid for b = 1 and 2 only. For higher values of b, $p(5^a7^b11^c + ë)$ is divisible not by $5^a7^b11^c$, but by $5^a7[^{(0+2)/2}]11^c$.

For b = 3, $\lambda = 243$

For b = 4, $\lambda = 2301$

For b = 5, $\lambda = 11905$

For b = 6, $\lambda = 112747$

Thus the next failure of Ramanujan's conjecture after p(243) will occur for p(2301) and obviously might not have been detected even now.

Thus, here Ramanujan, by his careful observation, intuition and insight had reached very near the truth. The need of reaching the truth led to further developments in mathematics.

This is also the mark of the work of a great mathematician that his work, even when it is not perfect leads to further developments in mathematics and sometimes these developments may be even more fruitful than his successes!

The discovery of the result that $24\lambda - 1$ should be divisible by $5^a7^b11^c$ showed a great and incredible insight. Ramanujan proved many other identities connected with the congruence properties of partitions e.g., one such result he proved was

p(4) + p(9)x + p(14)x^2 + ...

$$= 5\frac{\left[\left(1-x^{5}\right)\left(1-x^{10}\right)\left(1-x^{15}\right).......\right]^{5}}{\left[\left(1-x\right)\left(1-x^{2}\right)\left(1-x^{3}\right).......\right]^{6}}.$$

This result has been considered to be representative of the best of Ramanujan's work by Hardy, who says "If I had to select one formula from all Ramanujan's work, I would agree with Major Macmahen on selecting above".

Rogers – Ramanujan's Identities

We have given an example of Ramanujan's conjecture from partitions which was true upto n = 242 but had to be modified for larger values of n. We give now other examples of Ramanujan's conjectures for which he had done limited verification, but which were found to be true.

Ramanujan gave the identities:

$$1+\sum_{n\geq 1} q^{n^2} \Big/ (1-q)\left(1-q^{2}\right).........\left(1-q^{n}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-q^{5n+1}\right)^{-1}\left(1-q^{5n+1}\right)^{-1}$$

$$1+\sum_{n\geq 1} q^{n+n^2} \Big/ (1-q)\left(1-q^{2}\right).........\left(1-q^{n}\right)$$

$$= \prod_{n=0}^{\infty}\left(1-q^{5n+2}\right)^{-1}\left(1-q^{5n+3}\right)^{-1}$$

Ramanujan had verified that the first fifty terms or so of the power series expansion in q on both sides matched.

Ramanujan sent these identities to Hardy in 1913 and Major MacMahon verified these upto 89^{th} powers of q, but even these do not constitute a proof. Later when Ramanujan went to Cambridge, he found that Rogers had proved these in 1894 issue of Proceedings of London Mathematical Society, but these had been forgotten. Ramanujan's rediscovery of these brought fame to Rogers as well as the two together provided new proof in 1919. Meanwhile I. Schur of Germany proved these independently earlier in 1917 itself. Even more recently the Australian physicist R.J. Baxter rediscovered these while working on a problem of statistical mechanics. Still more recently the American mathematician Andrews and Baxter have given what they have called a motivated proof of these identities.

These identities illustrate the International nature of mathematics and the phenomenon of independent discovery in mathematics. Mathematician from India, England, Germany, Australia and America had independent motivations for proving these identities and the motivation came from consideration of Intuition, Rigour, Motivation, Partitions and Statistical Mechanics!

Concluding Remarks

(i) Ramanujan's contributions to theory of partition and Rogers–Ramanujan identities represent only a small part of his contributions of mathematics. We have chosen these topics because these can be relatively easily explained to the layman and these give us a flavour of Ramanujan's genius. These also illustrate not only Ramanujan's style of creative thinking, but of many lesser mathematicians as well.

(ii) Partition theory has many applications to physics and statistics, but Ramanujan's work is great not because it can have and does have applications, but because it shows the great height to which human genius can reach in meeting great intellectual challenges.

(iii) One of the criteria for measuring the stature of a mathematician is to see his impact on mathematics and mathematicians. Many mathematicians are forgotten even in their life time. Mathematics is developing so fast that 95% of what is created goes into oblivion. However, Ramanujan's work is having terrific impact even one hundred years after his birth. In fact his impact appears to increase every day. Hundreds of mathematicians are working on ideas initiated by him or inspired by him and thousands of papers are being written on his work. With each passing day his stature as a mathematician seems to grow.

(iv) Everybody cannot match his genius, but everybody can be inspired by his great dedication to mathematics, his great insight into mathematics, his hard work, his willingness to learn himself, his constant struggle for originality, his willingness to collaborate with others and his single-minded pursuit of mathematics.

(v) The best way to learn about mathematics and mathematicians is by attempting to solve some problems connected with the work. We give below some easy and some difficult problems:

(a) Find λ's for $0 \le a, b, c \le 5$

(b) Find the order of magnitude of $\sum_{r=1}^{n} r\ p(r)$

(c) Find all values of n below 10^4 for which p (n) will not satisfy Ramanujan's conjecture

(d) Find a formula for $\sum_{r=1}^{n} p(r)$

(e) Using (1) draw the graph of log p(n) against n and show that its slope approaches zero as n approaches infinity.

(f) Verify that Ramanujan's conjecture is true for all values of $n \le 100$.

(g) Verify that Roger-Ramanujan's identities are true up to the 20^{th} power of q.

(h) Show that the statement that n is not divisible by both 2^8 and 5^8 is true upto n=99, 999, 9999 and fails only after this value.

References

Andrews, G.E. 1987. The Lost Notebook and other unpublished writings of Srinivasa Ramanujan, Narosa Publishers.

Bharti Ramananda (Ed.). 1974. Professor Srinivasa Ramanujan Commemoration Volume, published by Prof. Srinivasa Ramanujan International Memorial Committee, Madras.

Hardy, G.H., P.V. Seshu Iyer and B.M. Wilson (Eds.). 1962. *Collected papers of Srinivasa Ramanujan*. Chalsea, New York.

________ 1940. Ramanujan, Chalsea, New York.

KAPUR, J.N. 1987. Ramanujan : A Personal Source of Inspiration, Presidential Address to Association of Mathematics Teachers Annual Conference.

________ Life and Works of Srinivasa Ramanujan, *Indian Journal of Mathematics Education*.

NAGARAJAN, K. R. AND T. SOUNDERARAJAN (Eds.). 1987. *Srinivasa Ramanujan*. (1887-1920). Macmillan India Limited.

Notebooks of SRINIVASA RAMANUJAN. 1957. 2 Volumes, Tata Institute of Fundamental Research.

RAMANUJAN'S Notebooks, Part I. 1985. Bruce C. Berndt, Spriger Verlag, New York.

RANGANATHAN, S. AND R. RAMANUJAN : *The Man and the Mathematician*. Asia Publishing House, Bombay.

SINGH, JAGJIT. 1987-1988. A short biography of Srinivasa Ramanujan, *Mathematical Education* (UGC Sponsored Journal), Vol. 4, Nos 1-4.

Special Issue on SRINIVASA RAMANUJAN, *Journal of Indian Institute of Science,* Bangalore.

SRINIVASAN, P.K. 1968(a). *Ramanujan's Letters and Reminiscences*. Memorial, Vol. 1, Muthialpet High School, Madras.

________ 1968(b). *Ramanujan : An Inspiration*. Memorial, Vol. 2, Muthialpet High School, Madras.

Appendix
Table of Values of p (n)

n	p(n)	n	p(n)	n	p(n)	n	p(n)
1	1	26	3436	51	239943	76	9289091
2	2	27	3010	52	281589	77	10619863
3	3	28	3718	53	329931	78	12132164
4	5	29	4565	54	386155	79	13848650
5	7	30	5604	55	451275	80	15796476
6	11	31	6842	56	526823	81	18004327
7	15	32	8349	57	614154	82	20506255
8	22	33	10143	58	715230	83	23338469
9	30	34	12310	59	831820	84	26543660
10	42	35	14883	60	966467	85	30167357
11	56	36	17977	61	1121505	86	34262962
12	77	37	21637	62	1300156	87	38887673
13	101	38	26015	63	1505499	88	44108109
14	135	39	31185	64	1741630	89	49995925
15	176	40	37338	65	2012558	90	56634173
16	231	41	44583	66	2323520	91	64112359
17	297	42	53174	67	2679689	92	72533807
18	385	43	63261	68	3087735	93	82010177
19	490	44	75175	69	3554345	94	92669720
20	627	45	89134	70	4087968	95	104651419
21	792	46	105558	71	4697205	96	118114304
22	1002	47	124754	72	5392783	97	133230930
23	1255	48	147273	73	6185689	98	150198136
24	1575	49	173525	74	7089500	99	169229875
25	1958	50	204226	75	8118264	100	190569292