DANGER OF IMPROPER USE OF MATHEMATICAL RESULTS, FORMULAS AND SYMBOLS: ALGEBRA

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Introduction

Mathematics Education has two aspects: the first deals with how to teach and the second with what to teach. The first aspect is concerned with the methodology of teaching mathematics, the "how and why" of learning process, the discussion of the objectives of learning mathematics, preparation of question bank, teachers' guides and so on. The second aspect deals mainly with the content, i.e., the subject matter of mathematics. In actual practice, teachers are often found to be giving more attention to the first aspect. It appears that the second aspect of mathematics education has not been paid so much attention resulting in students' getting wrong notions and concept in the content area. Since in teaching mathematics to the students, only manipulatory aspect has generally been emphasised, mathematical rules, formulas, theorems, etc., are mechanically memorised by the students without giving importance to the aspects – the language, the thought process and the logic of the subject. As a result, students sometimes get used to mathematical results, formulas, etc. without giving a thought to their validity and restrictions on them, if any, thus arriving at some fallacious results and mistakes for which they hardly find proper explanation.

In a series of articles of which this is the first, these types of improper use of results, formulas, etc., from various branches of mathematics and their consequences will be discussed with the help of examples.

Laws of indices

From the laws of indices, students know that if $a^x = a^y$, then $x = y$.

Also if $a^x = b^x$, then $a = b$.

Now, a student of even plus two stage may perhaps become puzzled and cannot explain the reason when he is asked the following questions:

Since $a^x = a^y$ $\quad x = y$,

Hence $1^1 = 1^2 = 1^3 = 1^4$ ....

$1 = 2 = 3 = 4 =$ ....

i.e., all numbers are equal – a result which is absurd. What is the reason of getting this absurd result?

Again, since $a^x = b^x$

$\Rightarrow$ a = b,

Hence $1^0 = 2^0 = 3^0 = 4^0 = \dots\dots = 1$

$\Rightarrow$ 1 = 2 = 3 = 4 = ...,

which is again an absurd result.

Why is this absurd result obtained?

Teachers generally don't mention or emphasise the restrictions involved in the above formulas of indices. In fact, these formulas are not true for all values of x, y or a, b. The formula $a^x = a^y \Rightarrow x = y$, for example, is valid when $a \neq 0$ and $a \neq 1$ and the formula $a^x = b^x \Rightarrow a = b$ is true only when $x \neq 0$. Most of the school students are not aware of these facts. We thus see that we get these absurd results due to the fact that we are not considering the limitations of the algebraic formulas. This is an example to show how the wrong use of a formula may lead one to get an absurd result.

Inequations

In solving an inequation of the type $x^2 \geq 1$, most of the students follow a wrong method to get the solution as $x \geq \pm 1$, which is wrong. Teacher generally don't emphasise this type of mistake. That $x \geq -1$ is wrong is obvious, since $x = 1/2$ (which is > -1) does not satisfy the inequation $x^2 \geq 1$. The correct solution will be $x \geq 1$ or $x \leq -1$, which can be obtained in the usual way of solving a quadratic inequation, i.e., by bringing 1 on the left side and factorising the expression on the left. Similarly, the solution of $x^2 \leq 1$ can be obtained in the above manner and not by merely taking the square root on both sides.

Sometimes students have the tendency to write ad > cb or bc > ad from the inequation a/b > c/d, without considering the signs of a, b, c and d. This is so because students are not generally warned by the teachers against this type of mistakes. In fact, we can write ad > cb, if b and d are of the same sign whatever be the signs of a and c. If, however, b and d are of opposite signs, then bc > ad is true. This is because we can multiply (hence divide) both sides of an inequation by a positive number without changing the signs of inequality and without affecting its solution.

The words "inequation" and "inequality" are sometimes wrongly treated as synonymous. The difference between these two is analogous to that between an equation and an identity. An inequation is satisfied by a particular set of values of the variables involved, whereas an inequality is true for all values of the variables. Thus $x + y \leq 1$ is an inequation, but $x^2 + y^2 \geq 2xy$ is an inequality. We "solve" an inequation whereas we "prove" an inequality. Thus, the usual question like "solve the inequality" is insignificant in the sense that every inequality involving a number of variables may have any arbitrary solution.

Modulus sign

Teachers sometimes give a wrong definition of modulus (absolute value) of a number. They generally explain the idea in the following way:

The modulus of a number means its positive value. Hence to get the modulus of a negative number, one should just remove the negative sign before it. The modulus of a number without any negative sign before it, is the number itself. Thus $|-2| = 2$, $|-x| = x$, $|x| = x$ etc. But the above concept is wrong in the sense that the rule is not applicable in the case of an algebraic number expressed by a variable. For example, if we assume that $|-x| = x$, then putting $x = -2$, we get $|2| = -2$, which is absurd. Again, if $|x| = x$, then also $|-2| = -2$ when $x = -2$. But this is also absurd. Thus, the correct and the most general definition of the modulus of any number, say x, is as follows:

$|x| = x$ if $x \quad 0$

$= -x$ if $x < 0$.

Logarithm

We sometimes use the formulas of logarithm without paying much attention to the restrictions involved in these formulas. As a result, we arrive at some absurd results due to this improper use of the formulas of logarithm.

For example, when we use the formula

$\log_b m^n = n \log_b m$

in some computation, we do not generally pay any attention about the signs of m, n and b.

Now, consider the following example:

Using the above formula, we have

$\log_{10} 16 = \log_{10} (-4)^2 = 2 \log_{10} (-4)$

Also, $\log_{10} 16 = \log_{10} 4^2 = 2 \log_{10} 4$

$\therefore \log_{10} (-4) = \log_{10} 4$

$\Rightarrow -4 = 4$

$\Rightarrow -1 = 1$

$\Rightarrow -2 = 2, -3 = 3, -4 = 4$, etc.

Thus, all positive numbers are equal to all negative numbers, which is obviously absurd. This absurd result is due to the improper use of the formula $\log_b m^n = n \log_b m$. It should be noted carefully that we can use this formula only when m (hence m^n) and b are positive and n is any real number.

Similarly, in the definition of $\log_n m$, we restrict m and n to be positive. If we are not careful about these restrictions, we may again come across with absurd results. For example,

$\log_{-1} 1 = 2$, since $(-1)^2 = 1$.

Also, $\log_{-1} 1 = 0$, since $(-1)^0 = 1$.

$\therefore 2 = 0$, which is absurd.

Consider another example:

We have $\log_{-1}(-1) = 3$, since $(-1)^3 = -1$.

Also, $\log_{-1}(-1) = 5$, since $(-1)^5 = -1$.

Hence $3 = 5$, which is again absurd. All these absurd results are due to the wrong use of the definition of $\log_b m$. In fact, the definition is based on the fact that both m and n must be positive.

In this connection, it can be stated that some students treat the numbers like $\overline{1}.235$ and -1.235 as synonymous. In simplifying an arithmetic expression with the help of a log table, a student may get an equation like $\log x = -1.782$ where x is the simplified value of the expression. He then sees antilog table for .782 and adjusts the decimal point to get the answer. But this procedure is completely wrong. The fact is that -1.782 and $\overline{1}.782$ are different. $\overline{1}.782$ means -1 + .782 i.e.,- .218 which is obviously different from -1.782.

The correct procedure to find x is, therefore, to make the mantissa part positive. The characteristic part may be positive or negative.

Thus $\log x = -1.782 = -2 + (2 - 1.782)$

$= -2 + .218 = \bar{2}.218$

Hence, to get x, we should see antilog for .218 and adjust the decimal point. This type of common mistakes by many students should be brought to their notice by the teachers.

Binomial Expansion

In the binomial expansion of $(1+x)^n$ where n is a negative integer or a fraction, we very often use the formula of the expansion without paying heed to the restrictions on x. We thus write

$$(1+x)^{-1} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots \text{ to } \infty$$

$$(1-x)^{-1} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \text{ to } \infty$$

Students memorise these formulas without thinking whether these formulas are true for all values of x or not. Now, observe the danger of the wrong use of the above-mentioned formulas

Putting x = 2 in the formula

$$(1-x)^{-1} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ to } \infty,$$

we get

$$(1-2)^{-1} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots \text{ to } \infty$$

$$\text{or, } -1 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots \text{ to } \infty$$

Left hand side is a negative number and the right hand side is a positive number and these two are equal, which is absurd. This absurd result is due to the fact that the above expansion formula is not valid for $|x| \geq 1$. In fact, when n is a negative integer or a fraction, the binomial expansion of $(1+x)^n$ is valid only when $|x| < 1$. This fact should be emphasized by the teachers in their classes.

Cancellation of a Common Factor from an Equation

In solving an algebraic equation, the natural tendency of most of the students is to cancel the common factor from both side without giving a thought whether the value of the factor may be zero or not. This type of wrong practice is very common among many students, because teachers generally do not emphasise the danger of dividing a number by zero. Cancellation of a common factor from both sides of an equation means dividing both sides by that factor. Now, if this factor is zero, then we are not allowed to divide by it, for division by zero is meaningless in mathematics.

Consider the following problem:

Solve completely the following equation:

$x^3 + x^2 - 4x = 0$

Students generally solve the equation as follows:

Dividing both sides of the equation by x, we get

$x^2 + x - 4 = 0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1+16}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{17}$$

But the above solutions are not the only solutions. x = 0 is also a solution of the given equation. Students will naturally miss this solution due to the fact that they have already divided both sides of the equation by x; in other words, they have unconsciously assumed the fact that $x \neq 0$ – an assumption which is not justified.

The correct method of solving the above equation is as follows:

From the given equation, we have

$x (x^2 + x - 4) = 0$

$\therefore$ Either $x = 0$ or, $x^2 + x - 4 = 0$.
The solution of $x^2 + x - 4 = 0$ has already been shown above.

Curious students may ask a natural question. If zero is treated to be a number, why is a student not allowed to divide a number by zero? To answer this question, teachers can give many examples to show that division by zero leads one to get absurd results in mathematics. One of many examples is as follows:

We know that the statement

$x^2 - x^2 = x^2 - x^2$

is always true.

The above statement implies

$x (x - x) = (x + x) (x - x)$

Dividing both sides by $x - x$, we get

$x = x + x$

or, $x = 2x$ or, $1 = 2$ (!)

This absurd result is due to division by $x - x$ which is zero as well as by x which may also be zero.

Extraneous Solution

In solving algebraic equations, students are satisfied if they get some solution of the equation by applying the usual methods. They hardly verify whether the solutions obtained satisfy the given equation/equations or not. As a result, they may sometimes get a wrong solution.

Consider, for example, the equation:

$$\sqrt{2x - 1} + \sqrt{x} = 2 \quad \text{..................} (1)$$

where positive values of the square roots should always be taken.

Squaring both sides of the equation (1), we get

$$3x - 1 + 2\sqrt{2x^2 - x} = 4$$

$$\Rightarrow 2\sqrt{2x^2 - x} = 5 - 3x$$

Squaring again, we get

$4 (2x^2 - x) = 25 + 9x^2 - 30 x$

$\Rightarrow \quad x^2 - 26 x + 25 = 0$
$\Rightarrow \quad (x - 25) (x - 1) = 0$
$\Rightarrow \quad x = 25$ or 1.
Hence, the solutions obtained should be $x = 25$ and $x = 1$. But if we put $x = 25$ in (1), we see that it does not satisfy this equation. Hence, $x = 25$ cannot be the solution and $x = 1$ is the only solution. The solution $x = 25$ which is called the extraneous solution should always be discarded. Teachers should make it clear to the students that this type of extraneous solution may be found in solving equation involving square roots and hence students should always verify the solutions by substituting these in the original equation.

Consistency and Inconsistency of Dependent Equations

Students know that a system of equations having no solution is inconsistent and if one equation is obtained from the other by multiplying by a constant, then the two equations are dependent. From this, they may conclude that two dependent equations are inconsistent, which is wrong. Teachers should emphasise the fact that two dependent equations in fact represent the same equation. These equations will have either one (in case of one variable) or infinite number (in case of more than one variable) or solution. Hence, two dependent equations are always consistent. On the other hand, two inconsistent (linear) equations involving two variables x and y represent two parallel lines which cannot meet together at a finite distance from the origin.

Determinants

In finding the value of a determinant of any order, students sometimes make wrong use of the operations with the elements of rows or columns of the determinant.

Students are familiar with the following theorems:

Theorem 1: The value of a determinant is unaltered if to each element of one column (or row) is added a constant multiple of the corresponding element of another column (or row). The following is the extension of the above theorem:
We can add multiples of any one column (or row) to every other column (or row) and leave the value of the determinant unaltered.

The practice of adding multiples of columns or rows at random is liable to lead to error unless each step is checked for validity by appeal to the following theorem:

Theorem 2: The determinant of order n

is equal to the sum of the 2^n determinants

$$\begin{vmatrix} a_1 + A_1 & b_1 + B_1 & \cdots\cdots & k_1 + K_1 \\ a_2 + A_2 & b_2 + B_2 & \cdots\cdots & k_2 + K_2 \\ \vdots & \vdots & & \vdots \\ a_n + A_n & b_n + B_n & \cdots\cdots & k_n + K_n \end{vmatrix}$$

corresponding to the 2^n different ways of choosing one letter from each column.

Now consider a third order determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

One of the curious errors into which one is led by adding multiples of rows at random is a fallacious proof that $\Delta = 0$ whatever a_i b_i c_i $(i = 1, 2, 3)$ may be. For example, in the above determinant, subtract the second column from the first, add the second and third columns and add the first and the third columns. We thus get

$$\Delta = \begin{vmatrix} a_1 - b_1 & b_1 + c_1 & c_1 + a_1 \\ a_2 - b_2 & b_2 + c_2 & c_2 + a_2 \\ a_3 - b_3 & b_3 + c_3 & c_3 + a_3 \end{vmatrix}$$

which is always zero. For,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix},$$

all other determinants being zero.

Hence,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Thus we have shown that whatever be the elements of the determinant, the value of the determinant is zero — a result which is absurd. We can also proceed in the following way to show this absurd result:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(Subtracting the elements in the second row from those in and then subtracting the elements is the first row from those in the second)

$$= - \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

= 0, the two rows being identical.

All these fallacious results are due to the fact that we have wrongly used the theorems stated above. In fact, when we add any multiple of a column (or row) to other column (or row), the former column (or row) should be left as it is. If, on the other hand, the former column (or row) is again added with a constant multiple of the latter column (or row) at the same time, then we shall get such fallacious results. The correct way of using the theorems stated above is as follows:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We can now add with the elements of the second row any constant multiple of the first row or second row and write the sum in the second row and leave the first or second row as it is.

Thus,

$$\Delta = \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_2 - a_3 & b_2 - b_3 & c_2 - c_3 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We can further write

$$\Delta = \begin{vmatrix} a_1 - a_2 & b_1 - b_2 & c_1 - c_2 \\ a_2 - a_3 & b_2 - b_3 & c_2 - c_3 \\ a_3 + a_2 & b_3 + b_2 & c_3 + c_2 \end{vmatrix}$$

This determinant cannot always be zero.

Improper Use of Symbols

It has been observed that in addition to improper use of mathematical results and formulas, students very often do mistakes in using some mathematical symbols and teachers also do not bother about these mistakes. As a result, students' concept about these symbols remain vague even when they study advanced mathematics. For example, they don't realise the difference between the symbols = and ⇒. While they simplify a mathematical expression, they use ⇒ in place of = and when they solve some equations or inequations, they use = in place of ⇒. It should be pointed out clearly to the students that we can use the symbol ⇒ between two statements and the symbol = between two expressions. An equation or an inequation or an identity (which has two sides) is an example of a statement.

Conclusion

The list of example which has been mentioned above is only a few out of many fallacious results which may arise due to the wrong use of formulas, results, etc., in Algebra. The main objective of mentioning these fallacious results is to show that mathematics is not a subject of mere memorisation, it needs a lot of thinking power and a very serious study. To have superficial knowledge in a particular topic in mathematics is more harmful than to have no knowledge in it. Our present day mathematics syllabus is so heavy and haphazard that it is very difficult even for teachers to have thorough concept in all topics in the present day school mathematics. We can, therefore, realise how dangerous it is for the students to have wrong concept in mathematics from the teachers.

With a view to remove some of the wrong concepts in different topics in mathematics which are now being taught in secondary and higher secondary classes from the minds of the teachers, a series of such papers each for a particular topic in mathematics will appear in the subsequent issues of this Journal. The author will be highly pleased if at least a few teachers are benefitted from the perusal of these papers.